# BIALGEBRA COHOMOLOGY, POINTED HOPF ALGEBRAS, AND DEFORMATIONS

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Abstract. We give explicit formulas for maps in a long exact sequence connecting bialgebra cohomology to Hochschild cohomology. We give a sufficient condition for the connecting homomorphism to be surjective. We apply these results to compute all bialgebra two-cocycles of certain Radford biproducts (bosonizations). These two-cocycles are precisely those associated to the finite dimensional pointed Hopf algebras in the recent classification of Andruskiewitsch and Schneider, in an interpretation of these Hopf algebras as graded bialgebra deformations of Radford biproducts.

## 1. Introduction

Gerstenhaber and Schack [8] found a long exact sequence connecting bialgebra cohomology to Hochschild cohomology. We reinterpret this sequence in the case of a finite dimensional Hopf algebra: Using results of Schauenburg [18] and Taillefer [20], bialgebra cohomology may be expressed as Hochschild cohomology of the Drinfeld double; we start by proving this directly in Section 3.2. Other terms in the long exact sequence (3.3.1) involve Hochschild cohomology, with trivial coefficients, of the Hopf algebra and of its dual. This version of the sequence is particularly useful for computation. We give an explicit description of the connecting homomorphism in formulas (3.4.1) and (3.4.2), and of the other two maps in the long exact sequence in Proposition 3.4.3 and formula (3.4.4). Theorem 4.2.3 gives a sufficient condition for the connecting homomorphism to be surjective in degree two.

As an application, we compute in Theorem 6.2.7 the (truncated) bialgebra cohomology, in degree two, of the finite dimensional graded pointed Hopf algebras arising in the recent classification of Andruskiewitsch and Schneider [2]. They classified all finite dimensional pointed Hopf algebras having abelian groups of grouplike elements, under a mild condition on the group orders. These include Lusztig's small quantum groups. In general they are filtered and are deformations (liftings) of their associated graded Hopf algebras, an observation of Du, Chen,

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and Ye [6] inspired by the graded algebraic deformation theory of Braverman and Gaitsgory [3]. These graded Hopf algebras are Radford biproducts, their deformations governed by bialgebra cohomology in degree two. In this setting, our Theorem 4.2.3 implies that the connecting homomorphism in the long exact sequence (3.3.1) is surjective (Theorem 6.2.1). We compute Hochschild cohomology with trivial coefficients (Theorem 6.1.4) and apply the connecting homomorphism to give the degree two bialgebra cohomology in Theorem 6.2.7. This computation is analogous to that of Grünenfelder and the first author [11] of cohomology associated to an abelian Singer pair of Hopf algebras. We describe all homogeneous bialgebra two-cocycles of negative degree. It follows from the classification of Andruskiewitsch and Schneider that they all lift to deformations, providing explicit examples for the Du-Chen-Ye theory. We give a further set of examples, the rank one pointed Hopf algebras of Krop and Radford [14], at the end of Section 6.

Our computation of cohomology gives insight into the possible deformations (liftings) of a Radford biproduct, providing a different way to see *why* the liftings of Andruskiewitsch and Schneider must look the way they do. In particular, we recover conditions for existence of certain relations in the Hopf algebra from those for existence of corresponding two-cocycles in Theorem 6.1.4 and equations (6.2.5) and (6.2.6). Our computational techniques may be useful in the search for pointed Hopf algebras that are left out of the classification, that is those having small prime divisors of the group order, complementing work of Heckenberger [12] on this problem. These techniques should also be helpful in understanding infinite dimensional pointed Hopf algebras. We will address these problems in future papers.

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## 2. Definitions and Preliminary Results

All vector spaces (algebras, coalgebras, bialgebras) will be over a ground field $k$. In the classification of Andruskiewitsch and Schneider, $k$ must be algebraically closed and of characteristic 0, however we do not require this for the general theory. If $A$ is an algebra and $C$ a coalgebra, then $\mathrm{Hom}_k(C, A)$ denotes the convolution algebra of all linear maps from $C$ to $A$. The unit and the multiplication on $A$ are denoted by $\eta\colon k \to A$ and $m\colon A \otimes A \to A$; the counit and the comultiplication on $C$ are denoted by $\varepsilon\colon C \to k$ and $\Delta\colon C \to C \otimes C$. We use Sweedler's notation for comultiplication: $\Delta(c) = c_1 \otimes c_2$, $(1 \otimes \Delta)\Delta(c) = c_1 \otimes c_2 \otimes c_3$, etc. If $f\colon U \otimes V \to W$ is a linear map, then we often write $f(u, v)$ instead of $f(u \otimes v)$. If $A$ is an augmented algebra, we denote the augmentation ideal by $A^+ = \ker \varepsilon$. If $V$ is a vector space, we denote its $n$-fold tensor power by $V^n$. To avoid confusion with comultiplication,

we write indices as superscripts, e.g. $v^1 \otimes \ldots \otimes v^n \in V^n$. If $A$ is an algebra, then we denote the $n$-ary multiplication by $\widehat{(-)}$, i.e. if $\mathbf{a} = a^1 \otimes \ldots \otimes a^n \in A^n$, then $\widehat{\mathbf{a}} = a^1 \cdots a^n$. If $C$ is a coalgebra, then $\Delta^n$ denotes the $n$-ary comultiplication, i.e. $\Delta^n c = c_1 \otimes \ldots \otimes c_n$.

2.1. **Bialgebra cohomology and deformations.** We recall the definition of bialgebra cohomology and its truncated version. For more details and greater generality we refer to [7]. Let $B$ be a bialgebra. The left and right diagonal actions and coactions of $B$ on $B^n$ will be denoted by $\lambda_l, \lambda_r, \rho_l, \rho_r$, respectively. More precisely, if $a \in B$ and $\mathbf{b} = b^1 \otimes \ldots \otimes b^n \in B^n$, then

$$\begin{aligned}
\lambda_l(a \otimes \mathbf{b}) &= a_1 b^1 \otimes \ldots \otimes a_n b^n,\\
\lambda_r(\mathbf{b} \otimes a) &= b^1 a_1 \otimes \ldots \otimes b^n a_n,\\
\rho_l(\mathbf{b}) &= \widehat{\mathbf{b}}_1 \otimes \mathbf{b}_2 = (b^1_1 \ldots b^n_1) \otimes (b^1_2 \otimes \ldots \otimes b^n_2),\\
\rho_r(\mathbf{b}) &= \mathbf{b}_1 \otimes \widehat{\mathbf{b}}_2 = (b^1_1 \otimes \ldots \otimes b^n_1) \otimes (b^1_2 \ldots b^n_2).
\end{aligned}$$

The standard complex for computing bialgebra cohomology is the following cosimplicial bicomplex $\mathbf{B}^{p,q}$. The vertices are $\mathbf{B}^{p,q} = \mathrm{Hom}_k(B^p, B^q)$. The horizontal faces

$$\partial^h_i \colon \mathrm{Hom}_k(B^p, B^q) \to \mathrm{Hom}_k(B^{p+1}, B^q)$$

and degeneracies

$$\sigma^h_i \colon \mathrm{Hom}_k(B^{p+1}, B^q) \to \mathrm{Hom}_k(B^p, B^q)$$

are those for computing Hochschild cohomology:

$$\begin{aligned}
\partial^h_0 f &= \lambda_l(1 \otimes f),\\
\partial^h_i f &= f(1 \otimes \ldots \otimes m \otimes \ldots \otimes 1),\ 1 \le i \le p,\\
\partial^h_{p+1} f &= \lambda_r(f \otimes 1),\\
\sigma^h_i f &= f(1 \otimes \ldots \otimes \eta \otimes \ldots \otimes 1);
\end{aligned}$$

the vertical faces

$$\partial^c_j \colon \mathrm{Hom}_k(B^p, B^q) \to \mathrm{Hom}_k(B^p, B^{q+1})$$

and degeneracies

$$\sigma^c_j \colon \mathrm{Hom}_k(B^p, B^{q+1}) \to \mathrm{Hom}_k(B^p, B^q)$$

are those for computing coalgebra (Cartier) cohomology:

$$\begin{aligned}
\partial^c_0 f &= (1 \otimes f)\rho_l,\\
\partial^c_j f &= (1 \otimes \ldots \otimes \Delta \otimes \ldots \otimes 1) f,\ \ 1 \le j \le q,\\
\partial^c_{q+1} f &= (f \otimes 1)\rho_r,\\
\sigma^c_i f &= (1 \otimes \ldots \otimes \varepsilon \otimes \ldots \otimes 1) f.
\end{aligned}$$

The vertical and horizontal differentials are given by the usual alternating sums

$$\partial^h = \sum (-1)^i \partial^h_i, \qquad \partial^c = \sum (-1)^j \partial^c_j.$$

The *bialgebra cohomology* of $B$ is then defined as

$$\mathrm{H}_b^*(B) = \mathrm{H}^*(\mathrm{Tot}\,\mathbf{B}).$$

where

$$\mathrm{Tot}\,\mathbf{B} = \mathbf{B}^{0,0} \to \mathbf{B}^{1,0} \oplus \mathbf{B}^{0,1} \to \ldots \to \bigoplus_{p+q=n} \mathbf{B}^{p,q} \xrightarrow{\partial^b} \ldots.$$

and $\partial^b$ is given by the sign trick (i.e., $\partial^b|_{\mathbf{B}^{p,q}} = \partial^h \oplus (-1)^1\partial^c \colon \mathbf{B}^{p,q} \to \mathbf{B}^{p+1,q} \oplus \mathbf{B}^{p,q+1}$). Here we abuse the notation by identifying a cosimplicial bicomplex with its associated cochain bicomplex. Let $\mathbf{B}_0$ denote the bicomplex obtained from $\mathbf{B}$ by replacing the edges by zeroes, that is $\mathbf{B}_0^{p,0} = 0 = \mathbf{B}_0^{0,q}$ for all $p, q$. The *truncated bialgebra cohomology* is

$$\widehat{\mathrm{H}}_b^*(B) = \mathrm{H}^{*+1}(\mathrm{Tot}\,\mathbf{B}_0).$$

For computations usually the normalized subcomplex $\mathbf{B}^+$ is used. The normalized complex $\mathbf{B}^+$ is obtained from the cochain complex $\mathbf{B}$ by replacing $\mathbf{B}^{p,q} = \mathrm{Hom}_k(B^p, B^q)$ with the intersection of degeneracies

$$(\mathbf{B}^+)^{p,q} = (\cap \ker \sigma_i^h) \cap (\cap \ker \sigma_j^c) \simeq \mathrm{Hom}_k((B^+)^p, (B^+)^q).$$

Note that we can identify

$$\widehat{\mathrm{H}}_b^1(B) = \{f \colon B^+ \to B^+ | f(ab) = af(b) + f(a)b,\ \Delta f(a) = a_1 \otimes f(a_2) + f(a_1) \otimes a_2\}$$

and

$$\widehat{\mathrm{H}}_b^2(B) = \widehat{\mathrm{Z}}_b^2(B)/\widehat{\mathrm{B}}_b^2(B),$$

where

$$\begin{aligned}
(2.1.1)\qquad \widehat{\mathrm{Z}}_b^2(B) \simeq \{(f,g)|\ & f\colon B^+ \otimes B^+ \to B^+,\ g\colon B^+ \to B^+ \otimes B^+,\\
& af(b,c) + f(a,bc) = f(ab,c) + f(a,b)c,\\
& c_1 \otimes g(c_2) + (1 \otimes \Delta)g(c) = (\Delta \otimes 1)g(c) + g(c_1) \otimes c_2,\\
& f(a_1,b_1) \otimes a_2b_2 - \Delta f(a,b) + a_1b_1 \otimes f(a_2,b_2) =\\
& \qquad -(\Delta a)g(b) + g(ab) - g(a)(\Delta b)\}
\end{aligned}$$

and

$$\begin{aligned}
(2.1.2)\qquad \widehat{\mathrm{B}}_b^2(B) \simeq \{(f,g)|\ & \exists h\colon B^+ \to B^+,\\
& f(a,b) = ah(b) - h(ab) + h(a)b\\
& g(c) = -c_1 \otimes h(c_2) + \Delta h(c) - h(c_1) \otimes c_2\}.
\end{aligned}$$

A *deformation* of the bialgebra $B$, over $k[t]$, consists of a $k[t]$-bilinear multiplication $m_t = m + tm_1 + t^2m_2 + \cdots$ and a comultiplication $\Delta_t = \Delta + t\Delta_1 + t^2\Delta_2 + \cdots$ with respect to which the $k[t]$-module $B[t] := B \otimes_k k[t]$ is again a bialgebra. In this paper, we are interested only in those deformations for which $\Delta_t = \Delta$, since the pointed Hopf algebras in the Andruskiewitsch-Schneider classification have this property. Given such a deformation of $B$, let $r$ be the smallest positive integer for

which $m_r \neq 0$ (if such an $r$ exists). Then $(m_r, 0)$ is a two-cocycle in $\widehat{\mathrm{Z}}^2_b(B)$. Every nontrivial deformation is equivalent to one for which the corresponding $(m_r, 0)$ represents a nontrivial cohomology class [7]. Conversely, given a positive integer $r$ and a two-cocycle $(m', 0)$ in $\widehat{\mathrm{Z}}^2_b(B)$, $m + t^r m'$ is an associative multiplication on $B[t]/(t^{r+1})$, making it into a bialgebra over $k[t]/(t^{r+1})$. There may or may not exist $m_{r+1}, m_{r+2}, \ldots$ for which $m + t^r m' + t^{r+1} m_{r+1} + t^{r+2} m^{r+2} + \ldots$ makes $B[t]$ into a bialgebra over $k[t]$. (For more details on deformations of bialgebras, see [7].)

2.2. **Graded bialgebra cohomology.** Here we recall the definition of graded (truncated) bialgebra cohomology [6]. If $B$ is a graded bialgebra, then $\mathbf{B}_{(l)}$ denotes the subcomplex of $\mathbf{B}$ consisting of homogeneous maps of degree $l$, more precisely

$$\mathbf{B}^{p,q}_{(l)} = \mathrm{Hom}_k(B^p, B^q)_l = \{f \colon B^p \to B^q | f \text{ is homogeneous of degree } l\}.$$

Complexes $(\mathbf{B}_0)_{(l)}$, $\mathbf{B}^+_{(l)}$ and $(\mathbf{B}^+_0)_{(l)}$ are defined analogously. The graded bialgebra and truncated graded bialgebra cohomologies are then defined by:

$$\begin{aligned} \mathrm{H}^*_b(B)_l &= \mathrm{H}^*(\mathrm{Tot}\,\mathbf{B}_{(l)}) = \mathrm{H}^*(\mathrm{Tot}\,\mathbf{B}^+_{(l)}), \\ \widehat{\mathrm{H}}^*_b(B)_l &= \mathrm{H}^{*+1}(\mathrm{Tot}(\mathbf{B}_0)_{(l)}) = \mathrm{H}^{*+1}(\mathrm{Tot}(\mathbf{B}^+_0)_{(l)}). \end{aligned}$$

Note that if $B$ is finite dimensional, then

$$\mathrm{H}^*_b(B) = \bigoplus_l \mathrm{H}^*_b(B)_l \quad \text{and} \quad \widehat{\mathrm{H}}^*_b(B) = \bigoplus_l \widehat{\mathrm{H}}^*_b(B)_l.$$

An $r$-*deformation* of $B$ is a bialgebra deformation of $B$ over $k[t]/(t^{r+1})$ given by $(m^r_t, \Delta^r_t)$. Given a graded bialgebra two-cocycle $(m', \Delta')$ of $B$, in degree $-r$, there exists an $r$-deformation, given by $(m + t^r m', \Delta + t^r \Delta')$. In this paper, we only consider $r$-deformations for which $\Delta^r_t = \Delta$.

**Remark 2.2.1.** (cf. [3, Prop. 1.5(c)], [10]) Suppose that $(B[t]/(t^r),\ m^{r-1}_t,\ \Delta^{r-1}_t)$ is an $(r-1)$-deformation, where

$$m^{r-1}_t = m + t m_1 + \ldots + t^{r-1} m_{r-1} \quad \text{and} \quad \Delta^{r-1}_t = \Delta + t\Delta_1 + \ldots + t^{r-1}\Delta_{r-1}.$$

If

$$D = (B[t]/(t^{r+1}), m^{r-1}_t + t^r m_r, \Delta^{r-1}_t + t^r \Delta_r)$$

is an $r$-deformation, then

$$D' = (B[t]/(t^{r+1}), m^{r-1}_t + t^r m'_r, \Delta^{r-1}_t + t^r \Delta'_r)$$

is an $r$-deformation if, and only if, $(m'_r - m_r, \Delta'_r - \Delta_r) \in \widehat{\mathrm{Z}}^2_b(B)_{-r}$. Note also that if $(m'_r - m_r, \Delta'_r - \Delta_r) \in \widehat{\mathrm{B}}^2_b(B)_{-r}$, then deformations $D$ and $D'$ are isomorphic.

2.3. **Coradically trivial and cotrivial cocycle pairs.** In this section we collect some preliminary results about cocycles that will be needed in Section 6. The first lemma largely follows from the theory of relative bialgebra cohomology [7]; however we did not find a proof in the literature and so we include one for completeness. Let $B$ be a graded bialgebra, and let $p\colon B \to B_0$ denote the canonical projection.

**Lemma 2.3.1.** *If* $\mathrm{char} k = 0$ *and* $B_0$ *is either a group algebra or the dual of a group algebra, then every* $(f,g) \in \widehat{\mathrm{Z}}^2_b(B)$ *is cohomologous to a cocycle pair* $(f',g')$ *for which* $f'|_{B_0\otimes B + B\otimes B_0} = 0$ *and* $(p\otimes 1)g' = 0 = (1\otimes p)g'$. *If* $f = 0$ *(resp.* $g = 0$*) then we can assume that also* $f' = 0$ *(resp.* $g' = 0$*).*

We say that $f'$ (respectively $g'$) is *trivial* (respectively *cotrivial*) on $B_0$ in case it satisfies the conclusion of the lemma.

*Proof.* Let $t \in B_0$ be the left and right integral in $B_0$ such that $\varepsilon(t) = 1$. Note also that $t_1 \otimes S(t_2) = S(t_1)\otimes t_2$. Recall that for $a \in B_0$ we have $t_1\otimes S(t_2)a = at_1\otimes S(t_2)$. We now proceed as follows.

**Step 1**: For each $f$, we will construct $s = s_f\colon B \to B$ such that

(1) $\partial^h(s)|_{B_0\otimes B} = f|_{B_0\otimes B}$.
(2) If $f|_{B\otimes B_0} = 0$, then $\partial^h(s)|_{B\otimes B_0} = 0$.
(3) If $g = 0$, then $\partial^c(s) = 0$.
(4) If $(p\otimes 1)g = 0$, then $(p\otimes 1)\partial^c(s) = 0$.
(5) If $(1\otimes p)g = 0$, then $(p\otimes 1)\partial^c(s) = 0$.

Define $s = s_f\colon B \to B$ by $s(b) = t_1 f(S(t_2), b)$. We claim that $s$ has the required properties:

(1) For $a \in B_0$ and $b \in B$ we compute

$$\begin{aligned}
(\partial^h s)(a,b) &= as(b) - s(ab) + s(a)b \\
&= at_1 f(S(t_2), b) - t_1 f(S(t_2), ab) + t_1 f(S(t_2), a)b \\
&= t_1 f(S(t_2)a, b) - t_1 f(S(t_2), ab) + t_1 f(S(t_2), a)b \\
&= t_1 S(t_2) f(a,b) - t_1 S(t_2) f(a,b) + t_1 f(S(t_2)a, b) \\
&\quad - t_1 f(S(t_2), ab) + t_1 f(S(t_2), a)b \\
&= f(a,b) - t_1\left(\partial^h f(S(t_2), a, b)\right) = f(a,b).
\end{aligned}$$

(2) If $f$ is such that $f|_{B\otimes B_0} = 0$, then $(\partial^h s)|_{B\otimes B_0} = 0$:

$$\begin{aligned}
(\partial^h s)(b,a) &= bs(a) - s(ba) + s(b)a \\
&= bt_1 f(S(t_2), a) - t_1 f(S(t_2), ba) + t_1 f(S(t_2), b)a \\
&= -t_1 f(S(t_2), ba) + t_1 f(S(t_2), b)a \\
&= -t_1 S(t_2) f(b,a) + t_1 f(S(t_2)b, a) + t_1(\partial^h f(S(t_2), b, a)) \\
&= 0.
\end{aligned}$$

(3)

$$\begin{aligned}(\partial^c s)(b) &= b_1 \otimes s(b_2) - \Delta s(b) + s(b_1) \otimes b_2\\ &= b_1 \otimes t_1 f(S(t_2), b_2) - \Delta t_1 f(S(t_2), b) + t_1 f(S(t_2), b_1) \otimes b_2\\ &= (t_1 \otimes t_2)(\partial^c f)(S(t_3), b)\\ &= -(t_1 \otimes t_2)(\partial^h g)(S(t_3), b).\end{aligned}$$

(4)

$$\begin{aligned}(p \otimes 1)(\partial^c s)(b) &= -(p \otimes 1)(t_1 \otimes t_2)(\partial^h g)(S(t_3), b)\\ &= -(t_1 \otimes t_2)\big[(p(S(t_4)b_1) \otimes S(t_3)b_2)(p \otimes 1)g(S(t_5)b_3)\\ &\quad -(p \otimes 1)g(S(t_3)b)\\ &\quad +(((p \otimes 1)g(S(t_5)b_1))(p(S(t_4)b_2) \otimes S(t_3)b_3))\big]\\ &= 0.\end{aligned}$$

(5) A symmetric version of the computation above works.

**Step 2.** Define $s' = s'_f \colon B \to B$ a right-hand side version of $s$ by $s'(b) = f(b, t_1)S(t_2)$ and note that $s'$ has properties analogous to those for $s$. Now define $r = r_f \colon B \to B$ by $r_f = s'_f + s_{f-\partial^h s'_f}$ and observe that $f_{B_0 \otimes B + B \otimes B_0} = \partial^h r_f$ and that $\partial^c r_f$ is $B_0$-cotrivial (resp. equal to 0) whenever $g$ is such.

**Step 3.** We dualize Step 2. Note that $g^* \colon B^* \otimes B^* = (B \otimes B)^* \to B^*$ is a Hochschild cocycle and $r_{g^*} \colon B^* \to B^*$ (see Step 2) is such that $\partial^h r_{g^*}|_{B_0^* \otimes B^* + B^* \otimes B_0^*} = g^*|_{B_0^* \otimes B^* + B^* \otimes B_0^*}$ and $\partial^c r_{g^*}$ is $B_0^*$-cotrivial (resp. equal to 0) whenever $f^*$ is $B_0^*$-cotrivial (resp. equal to 0). Now dualize again to obtain $u_g := r^*_{g^*} \colon B \to B$ and note that $g - \partial^c u_g$ is $B_0$-cotrivial and that $\partial^h u_g$ is $B_0$-trivial (resp. equal to 0) whenever $f$ is such.

**Step 4.** Define $v = v_{f,g} \colon B \to B$ by $v = u_g + s_{f-\partial^h u_g}$ and note that $(f', g') := (f, g) - (\partial^h v, \partial^c v)$ is a $B_0$-trivial, $B_0$-cotrivial cocycle pair. □

**Remark 2.3.2.** The above proof shows that the conclusion of the Lemma 2.3.1 holds whenever $B_0$ is either a commutative or cocommutative semisimple and cosemisimple Hopf algebra (with no assumptions on $k$).

**Remark 2.3.3.** If $B = R\#B_0$ as an algebra for some algebra $R$, and $f \colon B \otimes B \to B$ is a $B_0$-trivial Hochschild cocycle, then $f$ is uniquely determined by its values on $B^+ \otimes B^+$. More precisely, if $x, x' \in R$ and $h, h' \in B_0$, then $f(xh, x'h') = f(x, {}^h x')$.

**Definition 2.3.4.**

$$\widehat{\mathrm{Z}}^2_b(B)^+ = \left\{ (f, g) \in \widehat{\mathrm{Z}}^2_b(B) \,\middle|\, f \textit{ is } B_0\textit{-trivial}, \ g \textit{ is } B_0\textit{-cotrivial} \right\}.$$

If $f \colon B \otimes B \to B$, and $r$ is a nonnegative integer, then define $f_r \colon B \otimes B \to B$ by $f_r|_{(B\otimes B)_r} = f|_{(B\otimes B)_r}$ and $f_r|_{(B\otimes B)_s} = 0$ for $s \neq r$. If $g \colon B \to B \otimes B$, then we

define $g_r$ analogously. Note that $f = \sum_{r\geq 0} f_r$ and $g = \sum_{r\geq 0} g_r$. Define $f_{\leq r}$ by $f_{\leq r} = \sum_{0\leq i\leq r} f_i$ and then $f_{<r}$, $g_{\leq r}$, $g_{<r}$ in similar fashion.

We will need the following lemma.

**Lemma 2.3.5.** *Let $r$ be a positive integer and let $f\colon B\otimes B \to B$ be a homogeneous Hochschild cocycle (with respect to the left and right regular actions of $B$). If $f_{<r} = 0$, then $f_r\colon B \otimes B \to B$ is an $\varepsilon$-cocycle (i.e. a cocycle with respect to the trivial action of $B$ on $B$).*

*Proof.* We need to check that for homogeneous $x, y, z \in B$ with $deg(x)$, $deg(y)$, $deg(z) > 0$ we have $f_r(xy, z) = f_r(x, yz)$. Indeed, if $deg(x) + deg(y) + deg(z) \neq r$, then both sides are equal to 0. If $deg(x) + deg(y) + deg(z) = r$, then note that $f(x, y) = 0 = f(y, z)$ and hence $xf(y, z) - f(xy, z) + f(x, yz) - f(x, y)z = 0$ gives the desired conclusion. □

**Lemma 2.3.6.** *Assume $B$ is generated in degrees 0 and 1.*

(i) *If $(f, g) \in \widehat{\mathrm{Z}}^2_b(B)$, $r > 1$, $f_{\leq r} = 0$, and $g_{<r} = 0$, then $g_r = 0$.*
(ii) *If $(f, g) \in \widehat{\mathrm{Z}}^2_b(B)_l$, $l < -1$, $r > 0$, and $f_{\leq r} = 0$, then $g_{\leq r} = 0$.*
(iii) *If $(0, g) \in \widehat{\mathrm{Z}}^2_b(B)_l$ and $l < -1$, then $g = 0$.*
(iv) *If $(f, g) \in \widehat{\mathrm{Z}}^2_b(B)^+_l$, $l < 0$, $r > 0$, and $f_{\leq r} = 0$, then $g_{\leq r} = 0$.*
(v) *If $(0, g) \in \widehat{\mathrm{Z}}^2_b(B)^+_l$ and $l < 0$, then $g = 0$.*

*Proof.* (i) Note that $B_r$ is spanned by elements $xy$, where $x \in B_1$ and $y \in B_{r-1}$. Now observe $g_r(xy) = (\Delta x)g_r(y) + g_r(x)(\Delta y) - (\partial^c f)(x, y) = 0 + 0 - 0 = 0$.
(ii) Note the homogeneity of $g$ implies that $g_{\leq -l-1} = 0$. Hence if $r \leq -l + 1$ then we are done. If $r > -l + 1$, then use induction and part (i).
(iii) Follows from (ii).
(iv) Note that $B_0$-cotriviality of $g$ implies that $g_{\leq -l+1} = 0$. Hence if $r \leq -l + 1$ then we are done. If $r > -l + 1$, then use induction and part (i).
(v) Follows from (iv).

□

Using notation similar to that for bialgebra cohomology, we define the following in relation to Hochschild cohomology:

$$\begin{aligned}
\mathrm{Z}^2_h(B, B) &= \left\{ f\colon B \otimes B \to B \middle| \, \partial^h f = 0 \right\}, \\
\mathrm{B}^2_h(B, B) &= \left\{ \partial^h h \middle| \, h\colon B \to B \right\}, \\
\mathrm{H}^2_h(B, B) &= \mathrm{Z}^2_h(B, B)/\mathrm{B}^2_h(B, B), \\
\mathrm{Z}^2_h(B, B)_l &= \left\{ f \in \mathrm{Z}^2_h(B, B) \middle| \, f \text{ homogeneous of degree } l \right\}, \\
\mathrm{B}^2_h(B, B)_l &= \left\{ f \in \mathrm{B}^2_h(B, B) \middle| \, f \text{ homogeneous of degree } l \right\} \\
&= \left\{ \partial^h h \middle| \, h \text{ homogeneous of degree } l \right\}, \\
\mathrm{H}^2_h(B, B)_l &= \mathrm{Z}^2_h(B, B)_l/\mathrm{B}^2_h(B, B)_l.
\end{aligned}$$

Define $\pi\colon \widehat{\mathrm{Z}}^2_b(B) \to \mathrm{Z}^2_h(B)$, by $\pi(f,g) = f$. Note that $\pi$ maps subspaces $\widehat{\mathrm{B}}^2_b(B)$, $\widehat{\mathrm{Z}}^2_b(B)_l$, and $\widehat{\mathrm{B}}^2_b(B)_l$ into $\mathrm{B}^2_h(B)$, $\mathrm{Z}^2_h(B)_l$, and $\mathrm{B}^2_h(B)_l$ respectively. Hence $\pi$ gives rise to a map from $\widehat{\mathrm{H}}^2_b(B)$ to $\mathrm{H}^2_h(B,B)$ and a map from $\widehat{\mathrm{H}}^2_b(B)_l$ to $\mathrm{H}^2_h(B,B)_l$. We abuse notation by denoting these maps by $\pi$ as well.

We have the following relation between truncated bialgebra cohomology and Hochschild cohomology in degree two.

**Theorem 2.3.7.** *Assume $B$ is generated in degrees $0$ and $1$. If either $l < -1$ or $l = -1$ and $B_0$ is either a group algebra or a dual of a group algebra, then $\pi\colon \widehat{\mathrm{H}}^2_b(B)_l \to \mathrm{H}^2_h(B,B)_l$ is injective.*

*Proof.* Suppose $(f,g) \in \widehat{\mathrm{Z}}^2_b(B)_l$ represents a cohomology class in $\widehat{\mathrm{H}}^2_b(B)_l$ such that $f = \pi(f,g) \in \mathrm{B}^2_h(B,B)$. Note that if $l = -1$, then we can without loss of generality assume that $(f,g) \in \widehat{\mathrm{Z}}^2_b(B)^+_l$, hence $f = 0$ and thus by Lemma 2.3.6(v), also $g = 0$. Now assume that $l < -1$. Since $f$ is a Hochschild coboundary there is $s\colon B \to B$ such that $f = \partial^h s$ and hence $(f,g) \sim (f,g) - (\partial^h s, \partial^c s) = (0, g - \partial^c s)$. By Lemma 2.3.6(ii) this means that $g - \partial^c s = 0$ and hence $(f,g) \in \widehat{\mathrm{B}}^2_b(B)_l$. $\square$

2.4. **Cocycles stable under a group action.** In this section we explain how to identify cocycles stable under a group action with cocycles on a smash product with a group algebra; this identification is useful in explicit computations such as those in the last section. Let $R$ be a $k$-algebra with an action of a finite group $\Gamma$ by automorphisms. Let $R\#k\Gamma$ denote the corresponding *smash product algebra*, that is $R\#k\Gamma$ is a free left $R$-module with basis $\Gamma$ and algebra structure given by $(rg)(sh) = (r(g \cdot s))(gh)$ for all $r, s \in R$, $g, h \in G$. If the characteristic of $k$ does not divide the order of $\Gamma$, then

$$\mathrm{H}^*_h(R\#k\Gamma, k) \simeq \mathrm{H}^*_h(R,k)^\Gamma \tag{2.4.1}$$

(see for example [19, Cor. 3.4]). Let $B = R\#k\Gamma$, which need not be a bialgebra in this subsection. If $f\colon R \otimes R \to k$ is a $\Gamma$-stable cocycle, then the corresponding cocycle $\bar{f}\colon B \otimes B \to k$ is given by $\bar{f}(rg, r'g') = f(r, {}^g r')$ for all $r, r' \in R$, $g \in \Gamma$. (We will use the same notation $f$ in place of $\overline{f}$ for convenience). This observation is a special case of the following general lemma (cf. [4, Thm. 5.1]):

**Lemma 2.4.2.** *Let $f \in \mathrm{Hom}_k(R^n, k) \simeq \mathrm{Hom}_{R^e}(R^{n+2}, k)$ be a function representing an element of $\mathrm{H}^n_h(R,k)^\Gamma$ expressed in terms of the bar complex for $R$. The corresponding function $\overline{f} \in \mathrm{Hom}_k(B^n, k) \simeq \mathrm{Hom}_{B^e}(B^{n+2}, k)$ expressed in terms of the bar complex for $B$ is given by*

$$\overline{f}(a_1h_1 \otimes \cdots \otimes a_nh_n) = f(a_1 \otimes {}^{h_1}a_2 \otimes \cdots \otimes {}^{h_1\cdots h_{n-1}}a_n) \tag{2.4.3}$$

*for all $a_1, \ldots, a_n \in R$ and $h_1, \ldots, h_n \in \Gamma$.*

*Proof.* We sketch a proof for completeness; similar results appear in [4] and elsewhere for other choices of coefficients. Let $\mathcal{D} = \oplus_{g\in\Gamma} R^e(g \otimes g^{-1})$, a subalgebra of

$B^e$. We claim that the bar resolution for $B$ (as $B^e$-module) is induced from the $\mathcal{D}$-projective resolution of $R$,

$$\cdots \xrightarrow{\delta_3} \mathcal{D}_2 \xrightarrow{\delta_2} \mathcal{D}_1 \xrightarrow{\delta_1} \mathcal{D}_0 \xrightarrow{m} R \to 0, \tag{2.4.4}$$

where $\mathcal{D}_0 = \mathcal{D}$ and

$$\mathcal{D}_n = \mathrm{Span}_k\{a_0h_0 \otimes \cdots \otimes a_{n+1}h_{n+1} \mid a_i \in R,\ h_i \in \Gamma,\ h_0 \cdots h_{n+1} = 1\}$$

is a $\mathcal{D}$-submodule of $B^{\otimes(n+2)}$. Indeed, a map $B^e \otimes_{\mathcal{D}} \mathcal{D}_n \xrightarrow{\sim} B^{\otimes(n+2)}$ is given by

$$(b_{-1} \otimes b_{n+2}) \otimes (b_0 \otimes \cdots \otimes b_{n+1}) \mapsto b_{-1}b_0 \otimes b_1 \otimes \cdots \otimes b_n \otimes b_{n+1}b_{n+2},$$

and its inverse $\psi$ is

$$a_0h_0 \otimes a_1h_1 \otimes \cdots \otimes a_{n+1}h_{n+1} \mapsto (1 \otimes h_0 \cdots h_{n+1}) \otimes (a_0h_0 \otimes \cdots \otimes a_{n+1}h_n^{-1} \cdots h_0^{-1}),$$

for $a_i \in R$ and $h_i \in \Gamma$.

There is a map $\phi$ from (2.4.4) to the bar complex for $R$, as they are both $R^e$-projective resolutions of $R$,

$$a_0h_0 \otimes \cdots \otimes a_{n+1}h_{n+1} \mapsto a_0 \otimes {}^{h_0}a_1 \otimes {}^{h_0h_1}a_2 \otimes \cdots \otimes {}^{h_0\cdots h_n}a_{n+1}.$$

(See [4, (5.2)].) Applying these maps $\psi, \phi$ of complexes, together with the isomorphism $\mathrm{Hom}_{B^e}(B^e \otimes_{\mathcal{D}} \mathcal{D}_n, k) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_n, k)$, we have

$$\begin{aligned}
\overline{f}(a_1h_1 \otimes \cdots \otimes a_nh_n) &= \psi^*\phi^* f(1 \otimes a_1h_1 \otimes \cdots \otimes a_nh_n \otimes 1)\\
&= \phi^* f((1 \otimes h_1 \cdots h_n) \otimes (1 \otimes a_1h_1 \otimes \cdots \otimes a_nh_n \otimes h_n^{-1} \cdots h_1^{-1}))\\
&= \phi^* f(1 \otimes a_1h_1 \otimes \cdots \otimes a_nh_n \otimes h_n^{-1} \cdots h_1^{-1})\\
&= f(a_1 \otimes {}^{h_1}a_2 \otimes {}^{h_1h_2}a_3 \otimes \cdots \otimes {}^{h_1\cdots h_{n-1}}a_n),
\end{aligned}$$

since the image of $f$ is the trivial module $k$. □

## 3. A Long Exact Sequence for Bialgebra Cohomology

When we are dealing with a truncated double complex, a standard tool for computing its cohomology is a long exact sequence. More precisely, if $\mathbf{A}$ is a cochain bicomplex, $\mathbf{A}_0$ its truncated bicomplex and $\mathbf{A}_1$ its edge bicomplex, then the short exact sequence of cochain complexes

$$0 \to \mathrm{Tot}\,\mathbf{A}_0 \to \mathrm{Tot}\,\mathbf{A} \to \mathrm{Tot}\,\mathbf{A}_1 \to 0$$

gives rise to a long exact sequence of cohomologies:

$$\ldots \to \mathrm{H}^*(\mathrm{Tot}\,\mathbf{A}_0) \to \mathrm{H}^*(\mathrm{Tot}(\mathbf{A})) \to \mathrm{H}^*(\mathrm{Tot}(\mathbf{A}_1)) \xrightarrow{\delta} \mathrm{H}^{*+1}(\mathrm{Tot}(\mathbf{A}_0)) \to \ldots,$$

where the connecting homomorphism $\delta\colon \mathrm{H}^*(\mathrm{Tot}(\mathbf{A}_1)) \to \mathrm{H}^{*+1}(\mathrm{Tot}(\mathbf{A}_0))$ is induced by the differential. In the context of bialgebra cohomology this was already used in [7]. Furthermore, if $\mathbf{A}$ is a cosimplicial bicomplex, then by the Eilenberg-Zilber Theorem [21] (see [11, Appendix] for the cosimplicial version) we have $\mathrm{H}^*(\mathrm{Tot}\,\mathbf{A}) \simeq \mathrm{H}^*(\mathrm{Diag}\,\mathbf{A})$. If $\mathbf{A}$ is associated to a pair of (co)triples and a distributive law

between them, then the cohomology of $\operatorname{Diag}\mathbf{A}$ is the cohomology associated to the composed (co)triple. On the other hand, if the bicomplex $\mathbf{A}$ arises from some mixed distributive law then often one can, with some finiteness assumptions, use some duality to "unmix" the distributive law. This strategy worked remarkably well when dealing with cohomology associated to an abelian Singer pair of Hopf algebras [11] and can also be applied to truncated bialgebra cohomology. For the sake of simplicity we deal with this aspect of theory on the level of (co)simplicial bicomplexes and do not go into such generalities as (co)triples and distributive laws between them.

3.1. **"Unmixed" complex for computing bialgebra cohomology.** From now on assume that $B$ is a finite dimensional Hopf algebra. Let $X = (B^{\mathrm{op}})^* = (B^*)^{\mathrm{cop}}$. We will denote the usual pairing $X \otimes B \to k$ by $\langle \_|\_ \rangle$, i.e. if $x \in X$ and $a \in B$, then $\langle x|a\rangle = x(a)$. Note that $X$ and $B$ act on each other in the usual way (if $x \in X$ and $a \in B$, then the actions are denoted by $lax, x^a, lxa, a^x$):

$$\begin{aligned}\langle lax|b\rangle = \langle x|ba\rangle;\quad lax &= \langle x_1|a\rangle x_2,\\ \langle x^a|b\rangle = \langle x|ab\rangle;\quad x^a &= \langle x_2|a\rangle x_1,\\ \langle y|lxa\rangle = \langle yx|a\rangle;\quad lxa &= \langle x|a_2\rangle a_1,\\ \langle y|a^x\rangle = \langle xy|a\rangle;\quad a^x &= \langle x|a_1\rangle a_2.\end{aligned}$$

Observe that the diagonal actions of $B$ on $X^n$ and of $X$ on $B^n$ are given by

$$\begin{aligned}la\mathbf{x} &= \langle \widehat{\mathbf{x}}_1|a\rangle \mathbf{x}_2,\\ \mathbf{x}^a &= \langle \widehat{\mathbf{x}}_2|a\rangle \mathbf{x}_1,\\ lx\mathbf{a} &= \langle x|\widehat{\mathbf{a}}_2\rangle \mathbf{a}_1,\\ \mathbf{a}^x &= \langle x|\widehat{\mathbf{a}}_1\rangle \mathbf{a}_2.\end{aligned}$$

We use the natural isomorphism

$$\operatorname{Hom}_k(B^q, B^p) \simeq \operatorname{Hom}_k(X^p \otimes B^q, k),$$

given by identifying linear maps $f\colon X^p \otimes B^q \to k$ with linear maps $\bar f\colon B^q \to B^p$, by $f(\mathbf{x}\otimes \mathbf{b}) = \langle \mathbf{x}|\bar f(\mathbf{b})\rangle$, to obtain a cosimplicial bicomplex

$$\mathbf{C} = \big(\operatorname{Hom}_k(X^p \otimes B^q, k), (\partial^X)^*, (\partial^B)^*\big)$$

from the complex $\mathbf{B}$ defined in Section 2.1. The dual faces

$$\begin{aligned}\partial_i^B &= (\partial_i^B)^{p,q}\colon X^p \otimes B^{q+1} \to X^p \otimes B^q,\\ \partial_j^B &= (\partial_j^B)^{p,q}\colon X^{p+1} \otimes B^q \to X^p \otimes B^q\end{aligned}$$

are

$$\begin{aligned}
\partial_0^B(\mathbf{x},\mathbf{a}) &= (\mathbf{x}^{a^1}, a^2\otimes\cdots\otimes a^{q+1}) = \langle\widehat{\mathbf{x}}_2|a^1\rangle(\mathbf{x}_1, a^2\otimes\cdots\otimes a^{q+1}),\\
\partial_i^B(\mathbf{x},\mathbf{a}) &= (\mathbf{x}, a^1\otimes\cdots\otimes a^i a^{i+1}\otimes\cdots\otimes a^{q+1}),\ 1\le i\le q,\\
\partial_{q+1}^B(\mathbf{x},\mathbf{a}) &= (la^{q+1}\mathbf{x}, a^1\otimes\cdots\otimes a^q) = \langle\widehat{\mathbf{x}}_1|a^{q+1}\rangle(\mathbf{x}_2, a^1\otimes\cdots\otimes a^q),\\
\partial_0^X(\mathbf{x},\mathbf{a}) &= (x^2\otimes\cdots\otimes x^{p+1}, \mathbf{a}^{x^1}) = \langle x^1|\widehat{\mathbf{a}}_1\rangle(x^2\otimes\cdots\otimes x^{p+1}, \mathbf{a}_2)\\
\partial_j^X(\mathbf{x},\mathbf{a}) &= (x^1\otimes\cdots\otimes x^j x^{j+1}\otimes\cdots\otimes x^{p+1}, \mathbf{a}),\ 1\le j\le p,\\
\partial_{p+1}^X(\mathbf{x},\mathbf{a}) &= (x^1\otimes\cdots\otimes x^p, lx^{p+1}\mathbf{a}) = \langle x^{p+1}|\widehat{\mathbf{a}}_2\rangle(x^1\otimes\cdots\otimes x^p, \mathbf{a}_1).
\end{aligned}$$

The dual degeneracies

$$\begin{aligned}
\sigma_i^X &= (\sigma_i^X)^{p,q}\colon X^p\otimes B^q\to X^{p+1}\otimes B^q,\\
\sigma_j^B &= (\sigma_j^B)^{p,q}\colon X^p\otimes B^q\to X^p\otimes B^{q+1}
\end{aligned}$$

are given by

$$\begin{aligned}
\sigma_i^X(x^1\otimes\cdots\otimes x^p\otimes\mathbf{b}) = x^1\otimes\cdots\otimes x^i\otimes 1\otimes x^{i+1}\otimes\cdots\otimes x^p\otimes\mathbf{b},\\
\sigma_j^B(\mathbf{x}\otimes b^1\otimes\cdots\otimes b^q) = \mathbf{x}\otimes b^1\otimes\ldots\otimes b^j\otimes 1\otimes b^{j+1}\otimes\cdots\otimes b^q,
\end{aligned}$$

and the differentials

$$\partial^B\colon X^p\otimes B^{q+1}\to X^p\otimes B^q,\qquad \partial^X\colon X^{p+1}\otimes B^q\to X^p\otimes B^q$$

are given by the usual alternating sums, i.e.

$$\partial^B = \sum(-1)^i\partial_i^B,\qquad \partial^X = \sum(-1)^j\partial_j^X.$$

Note that by the cosimplicial version of the Eilenberg-Zilber Theorem we have $\mathrm{H}^*(\mathrm{Tot}(\mathbf{C}))\simeq\mathrm{H}^*(\mathrm{Diag}(\mathbf{C}))$.

3.2. **The diagonal complex and cohomology of the Drinfeld double.** Note that the differential

$$\partial_d = (\partial^d)^*\colon\ \mathrm{Hom}_k(X^n\otimes B^n, k)\to\mathrm{Hom}_k(X^{n+1}\otimes B^{n+1}, k)$$

in the diagonal complex $\mathrm{Diag}(\mathbf{C})$ is given by $(\partial^d)^n = \sum_{i=0}^{n+1}(-1)^k\partial_i^d$, where $\partial_i^d = \partial_i^X\partial_i^B$.

Recall that $D(B) = X\bowtie B$, the Drinfeld double of $B$, is $X\otimes B$ as coalgebra and the multiplication is given by

$$(x\bowtie a)(y\bowtie b) = x(la_1 y^{S^{-1}(a_3)})\bowtie a_2 b = \langle y_1|a_1\rangle\langle y_3|S^{-1}(a_3)\rangle xy_2\bowtie a_2 b.$$

The associated flip $c\colon B\otimes X\to X\otimes B$, is given by

$$c(a,x) = la_1 x^{S^{-1}(a_3)}\otimes a_2 = \langle x_1|a_1\rangle\langle x_3|S^{-1}(a_3)\rangle x_2\otimes a_2.$$

This map induces $c_{i,j}\colon B^i\otimes X^j\to X^j\otimes B^i$ and $\tilde{c}_n\colon (X\bowtie B)^n\to X^n\otimes B^n$ in the obvious way. Note that

$$c_{i,j}(\mathbf{a}\otimes\mathbf{x}) = \langle\widehat{\mathbf{x}}_1|\widehat{\mathbf{a}}_1\rangle\langle\widehat{\mathbf{x}}_3|S^{-1}(\widehat{\mathbf{a}}_3)\rangle\mathbf{x}_2\otimes\mathbf{a}_2.$$

Define also a map

$$\phi_n\colon X^n \otimes B^n \to X^n \otimes B^n$$

by

$$\phi_n(\mathbf{x}, \mathbf{a}) = \langle \widehat{\mathbf{x}}_1 | S^{-1}(\widehat{\mathbf{a}}_1)\rangle \mathbf{x}_2 \otimes \mathbf{a}_2.$$

The following identities are due to the fact that in order to compute $\tilde{c}_n$, we can apply $c$'s in arbitrary order.

$$\begin{aligned}
\tilde{c}_{n+1} &= (1 \otimes c_{1,n} \otimes 1)(1 \otimes \tilde{c}_n) \\
\tilde{c}_{n+1} &= (1 \otimes c_{n,1} \otimes 1)(\tilde{c}_n \otimes 1) \\
\tilde{c}_{i+j+1} &= (1 \otimes c_{i,j} \otimes 1)(1 \otimes c_{i,1} \otimes c_{1,j} \otimes 1)(\tilde{c}_i \otimes 1 \otimes \tilde{c}_j) \\
\tilde{c}_{i+j+2} &= (1 \otimes c_{i,j} \otimes 1)(1 \otimes c_{i,2} \otimes c_{2,j} \otimes 1)(\tilde{c}_i \otimes \tilde{c}_2 \otimes \tilde{c}_j)
\end{aligned}$$

Recall that the standard complex for computing $\mathrm{H}_h^*(D(B), k)$, Hochschild cohomology of $D(B)$ with trivial coefficients, is given by

$$\mathbf{D}: \quad \ldots \to \mathrm{Hom}_k(D(B)^n, k) \overset{(\partial^h)^*}{\to} \mathrm{Hom}_k(D(B)^{n+1}, k) \to \ldots$$

where $\partial^h = (\partial^h)^n = \sum_{i=0}^{n+1}(-1)^i \partial_i^h$ and

$$\begin{aligned}
\partial_0^h(u^1 \otimes \ldots \otimes u^{n+1}) &= \varepsilon(u^1)(u^2 \otimes \ldots \otimes u^{n+1}), \\
\partial_i^h(u^1 \otimes \ldots \otimes u^{n+1}) &= u^1 \otimes \ldots \otimes u^i u^{i+1} \otimes \ldots \otimes u^{n+1}, \\
\partial_{n+1}^h(u^1 \otimes \ldots \otimes u^{n+1}) &= \varepsilon(u^{n+1})(u^1 \otimes \ldots \otimes u^n).
\end{aligned}$$

**Theorem 3.2.1.** *The map $\phi_n \tilde{c}_n\colon (X \bowtie B)^n \to X^n \otimes B^n$ induces an isomorphism of complexes and hence $\mathrm{H}_b^*(B) \simeq \mathrm{H}^*(\mathrm{Diag}(\mathbf{C})) \simeq \mathrm{H}_h^*(D(B), k)$.*

*Proof.* Note that $\psi_n = \phi_n \tilde{c}_n$ is a linear isomorphism ( $\phi_n^{-1}(\mathbf{x}, \mathbf{a}) = \langle \widehat{\mathbf{x}}_1 | \widehat{\mathbf{a}}_1 \rangle \mathbf{x}_2 \otimes \mathbf{a}_2$, $c^{-1}(x, a) = \langle x_3 | a_3 \rangle \langle S^{-1}(x_1) | a_1 \rangle a_2 \otimes x_2$). We will show that for every $n$ and $0 \leq i \leq n+1$ the diagram

$$\begin{array}{ccc}
D(B)^{n+1} & \xrightarrow{\partial_i^h} & D(B)^n \\
{\scriptstyle \psi_{n+1}}\big\downarrow & & {\scriptstyle \psi_n}\big\downarrow \\
X^{n+1} \otimes B^{n+1} & \xrightarrow{\partial_i^d} & X^n \otimes B^n
\end{array}$$

commutes. We first deal with the case $i = 0$. Note that

$$\psi_n \partial_0^h = \phi_n \tilde{c}_n(\varepsilon \otimes 1) = (\varepsilon \otimes \phi_n)(1 \otimes \tilde{c}_n)$$

and that

$$\partial_0^d \psi_{n+1} = \partial_0^d \phi_{n+1}(1 \otimes c_{1,n} \otimes 1)(1 \otimes \tilde{c}_n).$$

Hence it is sufficient to prove that

$$(\varepsilon \otimes \phi_n) = (\partial_0^d)^n \phi_{n+1}(1 \otimes c_{1,n} \otimes 1).$$

This is achieved by the following computation

$$
\begin{aligned}
&[\partial_0^d \phi_{n+1}(1\otimes c_{1,n}\otimes 1)]((x\bowtie a)\otimes \mathbf{y}\otimes \mathbf{b})\\
&\quad = \partial_0^d \phi_{n+1}\langle \widehat{\mathbf{y}}_1|a_1\rangle\langle \widehat{\mathbf{y}}_3|S^{-1}(a_3)\rangle(x\otimes \mathbf{y}_2, a_2\otimes \mathbf{b})\\
&\quad = \partial_0^d \langle \widehat{\mathbf{y}}_1|a_1\rangle\langle \widehat{\mathbf{y}}_4|S^{-1}(a_4)\rangle\langle x_1\widehat{\mathbf{y}}_2|S^{-1}(a_2\widehat{\mathbf{b}}_1)\rangle(x_2\otimes \mathbf{y}_3, a_3\otimes \mathbf{b}_2)\\
&\quad = \langle \widehat{\mathbf{y}}_1|a_1\rangle\langle \widehat{\mathbf{y}}_5|S^{-1}(a_5)\rangle\langle x_1\widehat{\mathbf{y}}_2|S^{-1}(a_2\widehat{\mathbf{b}}_1)\rangle\langle x_3\widehat{\mathbf{y}}_4|a_3\rangle\langle x_2|a_4\widehat{\mathbf{b}}_2\rangle(\mathbf{y}_3,\mathbf{b}_3)\\
&\quad = \langle \widehat{\mathbf{y}}_1|a_1\rangle\langle \widehat{\mathbf{y}}_5|S^{-1}(a_6)\rangle\langle x_1|S^{-1}(a_3\widehat{\mathbf{b}}_2)\rangle\langle \widehat{\mathbf{y}}_2|S^{-1}(a_2\widehat{\mathbf{b}}_1)\rangle\\
&\qquad \cdot\langle x_3|a_4\rangle\langle \widehat{\mathbf{y}}_4|a_5\rangle\langle x_2|\widehat{\mathbf{b}}_3\rangle(\mathbf{y}_3,\mathbf{b}_4)\\
&\quad = \langle x|a_4\widehat{\mathbf{b}}_3S^{-1}(\widehat{\mathbf{b}}_2)S^{-1}(a_3)\rangle\langle \widehat{\mathbf{y}}_1|S^{-1}(\widehat{\mathbf{b}}_1)S^{-1}(a_2)a_1\rangle\langle \widehat{\mathbf{y}}_3|S^{-1}(a_6)a_5\rangle(\mathbf{y}_2,\mathbf{b}_4)\\
&\quad = \varepsilon(a)\varepsilon(x)\langle \widehat{\mathbf{y}}_1|S^{-1}(\widehat{\mathbf{b}}_1)\rangle(\mathbf{y}_2,\mathbf{b}_2)\\
&\quad = (\varepsilon\otimes\phi_n)((x\bowtie a)\otimes\mathbf{y}\otimes\mathbf{b}).
\end{aligned}
$$

A similar computation applies to $i=n+1$. The remaining cases, where $1\le i\le n$, are settled by the diagram below (where each of the squares is easily seen to commute).

$$
\begin{tikzcd}
D(B)^{i-1}\otimes D(B)^2\otimes D(B)^{n-i} \arrow[r, "\partial_i^h=1\otimes m\otimes 1"] \arrow[d, "\tilde{c}_{i-1}\otimes\tilde{c}_2\otimes\tilde{c}_{n-i}"'] & D(B)^{i-1}\otimes D(B)\otimes D(B)^{n-i} \arrow[d, "\tilde{c}_{i-1}\otimes 1\otimes\tilde{c}_{n-i}"'] \\
X^{i-1}\otimes B^{i-1}\otimes X^2\otimes B^2\otimes X^{n-i}\otimes B^{n-i} \arrow[r, "1\otimes 1\otimes m\otimes m\otimes 1\otimes 1"] \arrow[d, "1\otimes c_{i-1,2}\otimes c_{2,n-i}\otimes 1"'] & X^{i-1}\otimes B^{i-1}\otimes X\otimes B\otimes X^{n-i}\otimes B^{n-i} \arrow[d, "1\otimes c_{i-1,1}\otimes c_{1,n-i}\otimes 1"'] \\
X^{i+1}\otimes B^{i-1}\otimes X^{n-i}\otimes B^{n-i+2} \arrow[r, "(1\otimes m)\otimes 1\otimes 1\otimes (m\otimes 1)"] \arrow[d, "1\otimes c_{i-1,n-i}\otimes 1"'] & X^{i}\otimes B^{i-1}\otimes X^{n-i}\otimes B^{n} \arrow[d, "1\otimes c_{i-1,n-i}\otimes 1"'] \\
X^{n+1}\otimes B^{n+1} \arrow[r, "\partial_i^d=(1\otimes m\otimes 1)\otimes(1\otimes m\otimes 1)"] \arrow[d, "\phi_{n+1}"'] & X^{n}\otimes B^{n} \arrow[d, "\phi_n"'] \\
X^{n+1}\otimes B^{n+1} \arrow[r, "\partial_i^d"] & X^{n}\otimes B^{n}
\end{tikzcd}
$$

□

**Remark 3.2.2.** The isomorphism $\mathrm{H}_b^*(B)\simeq \mathrm{H}_h^*(D(B),k)$ can also be deduced from a result of Taillefer [20], combined with the fact due to Schauenburg [18] that the category of Yetter-Drinfeld modules is equivalent to the category of Hopf bimodules. See the remark following Proposition 4.6 in [20].

3.3. **Long exact sequence.** Let $\mathbf{C}_0$ denote the bicomplex obtained from $\mathbf{C}$ by replacing the edges by zeroes and let $\mathbf{C}_1$ denote the edge subcomplex of $\mathbf{C}$. Then we have a short exact sequence of bicomplexes

$$0\to \mathbf{C}_0\to\mathbf{C}\to\mathbf{C}_1\to 0,$$

hence a short exact sequence of their total complexes

$$0 \to \mathrm{Tot}(\mathbf{C}_0) \xrightarrow{\iota} \mathrm{Tot}(\mathbf{C}) \xrightarrow{\pi} \mathrm{Tot}(\mathbf{C}_1) \to 0,$$

which then gives rise to a long exact sequence of cohomologies ($i \geq 1$)

$$\dots \xrightarrow{\mathrm{H}(\iota)} \mathrm{H}^i(\mathrm{Tot}(\mathbf{C})) \xrightarrow{\mathrm{H}(\pi)} \mathrm{H}^i(\mathrm{Tot}(\mathbf{C}_1)) \xrightarrow{\delta} \mathrm{H}^{i+1}(\mathrm{Tot}(\mathbf{C}_0)) \to \dots$$

Now use isomorphisms

$$\begin{aligned}
&\mathrm{H}^i(\mathrm{Tot}(\mathbf{C}_1)) \simeq \mathrm{H}^i_h(B,k) \oplus \mathrm{H}^i_h(X,k),\\
&\mathrm{H}^i(\mathrm{Tot}(\mathbf{C})) \simeq \mathrm{H}^i(\mathrm{Diag}(\mathbf{C})) \simeq \mathrm{H}^i_h(X \bowtie B, k) \text{ and}\\
&\mathrm{H}^i(\mathrm{Tot}(\mathbf{C}_0)) \simeq \widehat{\mathrm{H}}^{i-1}_b(B),
\end{aligned}$$

to get a long exact sequence (cf. [8, §8])

$$\dots \xrightarrow{\bar{\iota}} \mathrm{H}^i_h(D(B),k) \xrightarrow{\bar{\pi}} \mathrm{H}^i_h(X,k) \oplus \mathrm{H}^i_h(B,k) \xrightarrow{\delta} \widehat{\mathrm{H}}^i_b(B) \to \dots \tag{3.3.1}$$

3.4. **Morphisms in the sequence.** Note that the morphism

$$\mathrm{H}^i_h(B,k) \oplus \mathrm{H}^i_h(X,k) \xrightarrow{\delta} \widehat{\mathrm{H}}^i_h(B)$$

corresponds to the connecting homomorphism in the long exact sequence and is therefore given by the differential, i.e. if $f\colon B^i \to k$ and $g\colon X^i \to k$ are cocycles, then $\delta(f,g) = (\partial^X f, (-1)^i \partial^B g)$. More precisely

$$F := \partial^X f \in \mathrm{Hom}_k(B^i, B) \subseteq \bigoplus_{m+n=i+1} \mathrm{Hom}_k(B^m, B^n),$$

is given by

$$F(\mathbf{b}) = f(\mathbf{b}_1)\widehat{\mathbf{b}_2} - f(\mathbf{b}_2)\widehat{\mathbf{b}_1}. \tag{3.4.1}$$

If we identify $g$ with an element of $B^i$ ($g \in (X^i)^* \simeq (B^i)^{**} \simeq B^i$), then

$$G := (-1)^i \partial^B g \in \mathrm{Hom}_k(B, B^i) \subseteq \bigoplus_{m+n=i+1} \mathrm{Hom}_k(B^m, B^n)$$

is given by

$$G(b) = (-1)^i \left((\Delta^i b)g - g(\Delta^i b)\right). \tag{3.4.2}$$

Recall that $\Delta^i b = b_1 \otimes \dots \otimes b_i$.

Using the cosimplicial Alexander-Whitney map, we can also show that the map $\overline{\pi}$ in the sequence (3.3.1) above is the double restriction:

**Proposition 3.4.3.** *The map*

$$\mathrm{H}^i_h(D(B),k) \xrightarrow{\bar{\pi}} \mathrm{H}^i_h(B,k) \oplus \mathrm{H}^i_h(X,k)$$

*is the restriction map in each component.*

*Proof.* We will establish the result by showing that the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Tot}^n(N\mathbf{C}) \arrow[r, "\Phi"] \arrow[d, equal] & \mathrm{Diag}^n(N\mathbf{C}) \arrow[r, "(\phi\tilde{c})^*"] & (D(B)^n)^* \arrow[d, "\mathrm{res}_2"'] \\
\mathrm{Tot}^n(N\mathbf{C}) \arrow[r, "\pi"] & \mathrm{Tot}^n(N\mathbf{C}_1) \arrow[r, "\subseteq"] & (X^n)^* \oplus (B^n)^*
\end{tikzcd}$$

Here $N\mathbf{C}$ denotes the normalized subcomplex of $\mathbf{C}$ (a map $f\colon X^p \otimes B^q \to k$ is in $N\mathbf{C}$ if $f(x^1 \otimes \ldots \otimes x^p, b^1 \otimes \ldots \otimes b^q) = 0$ whenever one of $x^i$ or $b^j$ is a scalar) and $\Phi$ denotes the Alexander-Whitney map (if $f \in (X^p \otimes B^q)^* \subseteq \oplus_{i+j=n}(X^i \otimes B^j)^*$, then $\Phi(f) \in (X^n \otimes B^n)^*$ is given by $\Phi(f) = f\partial^X_{p+1} \ldots \partial^X_n \partial^B_0 \ldots \partial^B_0$). Note that $\Phi(f)|_{X^n} = f(1_{X^p} \otimes \varepsilon_{X^{n-p}} \otimes \eta_{B^q})$ and that $\Phi(f)|_{B^n} = f(\eta_{X^p} \otimes \varepsilon_{B^{n-q}} \otimes 1_{B^q})$. Hence, if $f$ is normal, then

$$\Phi(f)|_{X^n} = \begin{cases} f; & p = n \\ 0; & p < n, \end{cases} \quad \text{and} \quad \Phi(f)|_{B^n} = \begin{cases} f; & q = n \\ 0; & q < n. \end{cases}$$

Also note that $\phi\tilde{c}|_{X^n} = 1_{X^n} \otimes \eta_{B^n}$ and $\phi\tilde{c}|_{B^n} = \eta_{X^n} \otimes 1_{B^n}$. Thus, if $\mathbf{f} = (f_0, \ldots, f_n) \in \bigoplus (X^i \otimes B^{n-i})^*$ is a normal cocycle, then $\mathrm{res}_{X^n}(\phi\tilde{c})^*\Phi\mathbf{f}(\mathbf{x}) = f_0(\mathbf{x})$ and $\mathrm{res}_{B^n}(\phi\tilde{c})^*\Phi(\mathbf{f})(\mathbf{b}) = f_n(\mathbf{b})$ and hence $\mathrm{res}_2(\phi\tilde{c})^*\Phi\mathbf{f} = (f_0, f_n) = \pi(\mathbf{f})$. $\square$

The map $\widehat{\mathrm{H}}^n_b(B) \xrightarrow{\bar{\iota}} \mathrm{H}^{n+1}(D(B), k)$ is given by the composite

$$\begin{aligned} \widehat{\mathrm{H}}^n_b(B) &\xrightarrow{\simeq} \mathrm{H}^{n+1}(\mathrm{Tot}(\mathbf{C}_1)) \xrightarrow{\iota} \mathrm{H}^{n+1}(\mathrm{Tot}(\mathbf{C})) \\ &\xrightarrow{\Phi} \mathrm{H}^{n+1}(\mathrm{Diag}(\mathbf{C})) \xrightarrow{(\phi\tilde{c})^*} \mathrm{H}^{n+1}(D(B), k). \end{aligned}$$

More precisely, if $\bar{f}\colon X^i \otimes B^{n+1-i} \to k$ corresponds to $f\colon B^{n+1-i} \to B^i$, then

$$\bar{\iota} f = \bar{f}\partial^X_{i+1} \ldots \partial^X_{n+1}\partial^B_0 \ldots \partial^B_0 \phi\tilde{c}. \tag{3.4.4}$$

3.5. **Graded version.** Now assume that $B$ is a finite dimensional graded Hopf algebra. Note that $X$ inherits the grading from $B$ and is nonpositively graded, and $D(B)$ is graded by both positive and negative integers. Note that morphisms in the long exact sequence preserve degrees of homogeneous maps and hence for every integer $l$ we get a long exact sequence:

$$\ldots \xrightarrow{\bar{\iota}} \mathrm{H}^i_h(D(B), k)_l \xrightarrow{\bar{\pi}} \mathrm{H}^i_h(X, k)_l \oplus \mathrm{H}^i_h(B, k)_l \xrightarrow{\delta} \widehat{\mathrm{H}}^i_b(B)_l \to \ldots$$

Also note that if $l$ is negative, then $\mathrm{H}^i_h(X, k)_l = 0$ (as $X$ is nonpositively graded and thus all homogeneous maps from $X$ to $k$ are of nonnegative degree), and hence in this case the sequence is

$$\ldots \xrightarrow{\bar{\iota}} \mathrm{H}^i_h(D(B), k)_l \xrightarrow{\bar{\pi}} \mathrm{H}^i_h(B, k)_l \xrightarrow{\delta} \widehat{\mathrm{H}}^i_b(B)_l \to \ldots$$

## 4. A sufficient condition for surjectivity of the connecting homomorphism

In this section, we give a sufficient condition for surjectivity of the connecting homomorphism $\delta$ in degree 2 of the long exact sequence (3.3.1). The surjectivity will allow us to compute fully the bialgebra cohomology in degree 2 for some general classes of examples in the last section.

4.1. **Second Hochschild cohomology of a graded Hopf algebra with trivial coefficients.** If $U \xrightarrow{f} V \xrightarrow{g} W$ is a sequence of vector space maps such $gf = 0$, then

$$\frac{\ker f^*}{\operatorname{im} g^*} \simeq \left(\frac{\ker g}{\operatorname{im} f}\right)^* \simeq \left[\ker\left(\tilde{g}\colon \frac{V}{\operatorname{im} f} \to W\right)\right]^*,$$

where $\tilde{g}$ is the map induced by $g$. We apply this observation to an augmented algebra $R$ with augmentation ideal $R^+$ and the map

$$R^+ \otimes R^+ \otimes R^+ \xrightarrow{m\otimes 1 - 1\otimes m} R^+ \otimes R^+ \xrightarrow{m} R^+$$

to compute Hochschild cohomology of $R$ with trivial coefficients:

$$\mathrm{H}^2_h(R,k) \simeq \frac{\ker(m\otimes 1 - 1\otimes m)^*}{\operatorname{im}(m^*)} \simeq \left[\ker\left(\tilde{m}\colon R^+ \otimes_{R^+} R^+ \to R^+\right)\right]^*,$$

where we abbreviate $R^+ \otimes_{R^+} R^+ = \frac{R^+\otimes R^+}{\operatorname{im}(m\otimes 1 - 1\otimes m)}$ and $\tilde{m}$ is the map induced by multiplication $m\colon R^+ \otimes R^+ \to R^+$. Also abbreviate

$$M := \ker\left(\tilde{m}\colon R^+ \otimes_{R^+} R^+ \to R^+\right). \tag{4.1.1}$$

The isomorphism above can be described explicitly as follows. Choose $\phi\colon (R^+)^2 = \operatorname{Span}\{xy | x, y \in R^+\} \to R^+ \otimes R^+$ a splitting of $m$. If we are given a linear map $g\colon M \to k$, then define a cocycle $\bar{g}\colon R^+ \otimes R^+ \to k$ by $f = g(id - \phi m)$. If $f\colon R^+ \otimes R^+ \to k$ is a cocycle, then $\tilde{f}\colon M \to k$ is simply the induced map. It is easy to check that $\tilde{\bar{g}} = g$ and that $\bar{\tilde{f}} = f - \partial^h(f\phi) \sim f$.

4.2. **Surjectivity of the connecting homomorphism.** If $B = \bigoplus_{n\geq 0} B_n$ is a graded Hopf algebra, and $p\colon B \to B_0$ is the canonical projection then $B$ equipped with $B_0 \overset{p}{\leftrightarrows} B$ is Hopf algebra with a projection in the sense of [17] and hence $R = B^{coB_0} = \{r \in B | (1\otimes p)\Delta r = r \otimes 1\}$ is a Hopf algebra in the category of Yetter-Drinfeld modules over $B_0$. The action of $B_0$ on $R$ is given by ${}^h r = h_1 r S(h_2)$ and coaction by $r \mapsto (p\otimes 1)\Delta r$. Throughout this section we assume that $B_0 = k\Gamma$ is a group algebra and that the action of the group $\Gamma$ on $R$ is diagonal. In this case $R$ is $(\Gamma \times \hat{\Gamma} \times \mathbb{N})$-graded, that is it decomposes as $R = \bigoplus R_{g,\chi,n}$, where $R_{g,\chi,n}$ consists of homogeneous elements $r \in R$ of degree $n$ such that the coaction of $k\Gamma$ is given by $r \mapsto g \otimes r$ and the action of $k\Gamma$ is given by ${}^h r = \chi(h) r$. We abbreviate $R_{g,l} = \bigoplus_{\chi\in\hat{\Gamma}} R_{g,\chi,l}$.

Observe that $(m \otimes 1 - 1 \otimes m)\colon R^+ \otimes R^+ \otimes R^+ \to R^+ \otimes R^+$ preserves the $(\Gamma \times \hat{\Gamma} \times \mathbb{N})$-grading and hence we can decompose $M$ (see (4.1.1)) in the same fashion:

$$M = \bigoplus_{(g,\chi,l) \in \Gamma \times \widehat{\Gamma} \times \mathbb{Z}_{\geq 2}} M_{g,\chi,l}, \tag{4.2.1}$$

where $M_{g,\chi,l}$ consists of homogeneous elements $m \in M$ of degree $l$ for which the action and coaction of $k\Gamma$ are given by ${}^h m = \chi(h)m$ and $m \mapsto g \otimes m$. Also note that $\mathrm{H}^2_h(B,k) = \mathrm{H}^2_h(R,k)^\Gamma \simeq \bigoplus_{(g,n)\in\Gamma\times\mathbb{Z}_{\geq 2}} M^*_{g,\varepsilon,n}$ and that if $V$ is a finite-dimensional trivial $B$-bimodule, then

$$\mathrm{H}^2_h(B,V) = \bigoplus_{(g,n)\in\Gamma\times\mathbb{Z}_{\geq 2}} \mathrm{Hom}_k(M_{g,\varepsilon,n}, V) = \bigoplus_{(g,n)\in\Gamma\times\mathbb{Z}_{\geq 2}} M^*_{g,\varepsilon,n} \otimes V.$$

The following lemma will be crucial in establishing a sufficient condition for surjectivity of the connecting homomorphism.

**Lemma 4.2.2.** *Let $B$ be a graded Hopf algebra of the form described at the beginning of this section and let $l < 0$. Assume also that whenever $M_{h,\varepsilon,j} \neq 0$ for some $h \in \Gamma$ and $j > -l$, then $B$ contains no nonzero $(1,h)$-primitive elements in degree $j + l$. If $R$ is generated as an algebra by $R_1$, then for any $(f,g) \in \widehat{\mathrm{Z}}^2_b(B)^+_l$ the following holds:*

(i) *If $r > -l$ and $f_{<r} = 0$, then $(f,g)$ is cohomologous to $(f',g') \in \widehat{\mathrm{Z}}^2_b(B)^+_l$, where $f'_{\leq r} = 0$.*

(ii) *If $f_{-l} = 0$, then $(f,g) \in \widehat{\mathrm{B}}^2_b(B)_l$.*

*Proof.* (i) Note that by Lemma 2.3.6(iv) we have $g_{<r} = 0$. By Lemma 2.3.5, $f_r$ is an $\varepsilon$-cocycle. If $u \in R^+ \otimes R^+$ represents an element in $M_{h,\chi,r} \neq 0$, then, considering (2.1.1), we have

$$0 = g_r(0) = g_r(m(u)) = h \otimes f_r(u) - \Delta f_r(u) + f_r(u) \otimes 1.$$

Hence $f_r(u) = 0$, since it is a $(1,h)$-primitive element of degree $r + l$. Since $f_r(u) = 0$ for all $u \in M$, we can conclude, due to the discussion above, that $f_r$ is an $\varepsilon$-coboundary. Thus we may let $s\colon B \to B$ be such that $f_r(x,y) = s(xy)$ for $x, y \in B^+$. Note that $\partial^h s$ is $B_0$-trivial, $\partial^c s$ is $B_0$-cotrivial and that $(f',g') = (f,g) - (\partial^h s, \partial^c s) \in \widehat{\mathrm{Z}}^2_b(B)^2_l$ is such that $f'_{\leq r} = 0$.

(ii) Use induction and part (i) to show that $(f,g) \sim (0,g')$. Then use Lemma 2.3.6(v). □

The following is one of the main theorems in our paper. The results in the rest of the paper rely heavily on it. Recall the notation defined in (4.2.1).

**Theorem 4.2.3.** *Suppose that $l < 0$ and $B$ is a finite dimensional graded Hopf algebra such that*

- *$B$ is generated as an algebra by $B_0$ and $B_1$.*
- *$B_0 = k\Gamma$ and the action of $B_0$ on $R$ is diagonalizable, i.e. $\Gamma$ acts on $R$ by characters.*
- *If $M_{h,\varepsilon,j} \neq 0$ for some $h \in \Gamma$ and $j > -l$, then $B$ contains no nonzero $(1,h)$-primitive elements in degree $j+l$.*

*Then the connecting homomorphism $\delta\colon \mathrm{H}^2_h(B,k)_l \to \widehat{\mathrm{H}}^2_b(B)_l$ is surjective.*

*Proof.* Let $(f,g) \in \widehat{\mathrm{Z}}^2_b(B)^+_l$. Note that $f_{\leq -l-1} = 0$ and that $g_{\leq -l+1} = 0$. Define $\tilde{f}\colon B \otimes B \to k$ by $\tilde{f}(a,b) = -p_1 f_{-l}(a,b)$, where $p_1\colon k\Gamma \to k$ is given by $p_1(g) = \delta_{1,g}$. Note that $\tilde{f}$ is an $\varepsilon$-cocycle by Lemma 2.3.5. Now we prove that $(f',g') = (f,g) - \delta\tilde{f} = (f,g) - (\partial^c\tilde{f}, 0) \in \widehat{\mathrm{B}}^2_b(B)_l$: This will follow from Lemma 4.2.2(ii) once we see that $f'_{-l} = 0$. Indeed, if $x \in R_{h_x,i}$, $y \in R_{h_y,-l-i}$, then $f(x,y) \in B_0$ is $(h_xh_y,1)$-primitive by the same argument as in the proof of Lemma 4.2.2(i). If $h_xh_y = 1$, then $f(x,y)$ is primitive and hence 0. Otherwise $f(x,y) = a(h_xh_y - 1)$ for some $a \in k$. Note that $a = -p_1(a(h_xh_y-1)) = -p_1(f(x,y)) = \tilde{f}(x,y)$ and that $\partial^c\tilde{f}(x,y) = \tilde{f}(x_2,y_2)x_1y_1 - \tilde{f}(x_1,y_1)x_2y_2 = \tilde{f}(x,y)(h_xh_y - 1) = f(x,y)$. Thus $f'_{-l} = f_{-l} - \partial^c\tilde{f} = 0$. □

**Remark 4.2.4.** If all $(1,h)$-primitive elements of $B$ in positive degree are contained in $R_1$ (this happens whenever $B$ is coradically graded and $B_0 = k\Gamma$), then it is sufficient to demand that there are no $(1,h)$-primitive elements in $R_1$, for all $h \in \Gamma$ for which $M_{h,\varepsilon,l+1} \neq 0$.

## 5. Finite dimensional pointed Hopf algebras

We recall the Hopf algebras of Andruskiewitsch and Schneider [2], to which we will apply the results of the previous sections.

Let $\theta$ be a positive integer. Let $(a_{ij})_{1\leq i,j\leq\theta}$ be a *Cartan matrix of finite type*, that is the Dynkin diagram of $(a_{ij})$ is a disjoint union of copies of the diagrams $A_\bullet, B_\bullet, C_\bullet, D_\bullet, E_6, E_7, E_8, F_4, G_2$. In particular, $a_{ii} = 2$ for $1 \leq i \leq \theta$, $a_{ij}$ is a nonpositive integer for $i \neq j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. Its Dynkin diagram is a graph with vertices labelled $1,\dots,\theta$. If $|a_{ij}| \geq |a_{ji}|$, vertices $i$ and $j$ are connected by $|a_{ij}|$ lines, and these lines are equipped with an arrow pointed toward $j$ if $|a_{ij}| > 1$.

Let $\Gamma$ be a finite abelian group. Let

$$\mathcal{D} = \mathcal{D}(\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta})$$

be a *datum of finite Cartan type* associated to $\Gamma$ and $(a_{ij})$; that is $g_i \in \Gamma$ and $\chi_i \in \widehat{\Gamma}$ $(1 \leq i \leq \theta)$ such that $\chi_i(g_i) \neq 1$ $(1 \leq i \leq \theta)$ and the Cartan condition

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}} \tag{5.0.5}$$

holds for $1 \leq i,j \leq \theta$.

Let $\Phi$ denote the root system corresponding to $(a_{ij})$, and fix a set of simple roots $\Pi$. If $\alpha_i, \alpha_j \in \Pi$, write $i \sim j$ if the corresponding nodes in the Dynkin diagram of $\Phi$ are in the same connected component. Choose scalars $\lambda = (\lambda_{ij})_{1\leq i<j\leq\theta,\ i\not\sim j}$, called *linking parameters*, such that

$$\lambda_{ij} = 0 \quad \text{if} \quad g_i g_j = 1 \quad \text{or} \quad \chi_i\chi_j \neq \varepsilon, \tag{5.0.6}$$

where $\varepsilon$ is the trivial character defined by $\varepsilon(g) = 1$ $(g \in \Gamma)$. Sometimes we use the notation

$$\lambda_{ji} := -\chi_i(g_j)\lambda_{ij} \quad (i < j). \tag{5.0.7}$$

The (infinite dimensional) Hopf algebra $U(\mathcal{D}, \lambda)$ defined by Andruskiewitsch and Schneider [2] is generated as an algebra by $\Gamma$ and symbols $x_1, \ldots, x_\theta$, subject to the following relations. Let $V$ be the vector space with basis $x_1, \ldots, x_\theta$. The choice of characters $\chi_i$ gives an action of $\Gamma$ by automorphisms on the tensor algebra $T(V)$, in which $g(x_{i_1}\cdots x_{i_s}) = \chi_{i_1}(g)\cdots\chi_{i_s}(g)x_{i_1}\cdots x_{i_s}$ $(g \in \Gamma)$. We use this action to define the braided commutators

$$\mathrm{ad}_c(x_i)(y) = [x_i, y]_c := x_i y - g_i(y)x_i,$$

for all $y \in T(V)$. The map $c\colon T(V)\otimes T(V) \to T(V)\otimes T(V)$, induced by $c(x_i\otimes y) = g_i(y) \otimes x_i$ is a braiding and $T(V)$ is a braided Hopf algebra in the Yetter-Drinfeld category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. (See [2] for details, however we will not need to use the theory of Yetter-Drinfeld categories.) There is a similar adjoint action $\mathrm{ad}_c$ on any quotient of $T(V)$ by a homogeneous ideal. The relations of $U(\mathcal{D}, \lambda)$ are those of $\Gamma$ and

$$gx_ig^{-1} = \chi_i(g)x_i \qquad (g \in \Gamma, 1 \leq i \leq \theta), \tag{5.0.8}$$

$$(\mathrm{ad}_c(x_i))^{1-a_{ij}}(x_j) = 0 \qquad (i \neq j,\ i \sim j), \tag{5.0.9}$$

$$(\mathrm{ad}_c(x_i))(x_j) = \lambda_{ij}(1 - g_ig_j) \qquad (i < j,\ i \not\sim j). \tag{5.0.10}$$

The coalgebra structure of $U(\mathcal{D}, \lambda)$ is defined by

$$\Delta(g) = g \otimes g, \quad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i,$$

for all $g \in \Gamma$, $1 \leq i \leq \theta$.

Let $W$ be the Weyl group of the root system $\Phi$. Let $w_0 = s_{i_1}\cdots s_{i_p}$ be a reduced decomposition of the longest element $w_0 \in W$ as a product of simple reflections. Let

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \ \cdots, \ \beta_p = s_{i_1}s_{i_2}\cdots s_{i_{p-1}}(\alpha_{i_p}).$$

Then $\beta_1, \ldots, \beta_p$ are precisely the positive roots $\Phi^+$. Corresponding root vectors $x_{\beta_j} \in U(\mathcal{D}, \lambda)$ are defined in the same way as for the traditional quantum groups: In case $\mathcal{D}$ corresponds to the data for a quantum group $U_q(\mathfrak{g})$, let

$$x_{\beta_j} = T_{i_1}T_{i_2}\cdots T_{i_{j-1}}(x_{i_j}),$$

where the $T_{i_j}$ are Lusztig's algebra automorphisms of $U_q(\mathfrak{g})$ [15]. In particular, if $\beta_j$ is a simple root $\alpha_l$, then $x_{\beta_j} = x_l$. The $x_{\beta_j}$ are in fact iterated braided

commutators. In our more general setting, as in [2], define the $x_{\beta_j}$ to be the analogous iterated braided commutators.

The Hopf algebra $U(\mathcal{D},\lambda)$ has the following finite dimensional quotients. As in [2] we make the assumptions:

$$\begin{aligned}&\textit{the order of } \chi_i(g_i) \textit{ is odd for all } i,\\ &\textit{and is prime to 3 for all } i \textit{ in a connected component of type } G_2.\end{aligned} \tag{5.0.11}$$

It follows that the order of $\chi_i(g_i)$ is constant in each connected component $J$ of the Dynkin diagram [2]; denote this common order by $N_J$. It will also be convenient to denote it by $N_{\alpha_i}$ or more generally by $N_{\beta_j}$ or $N_j$ for some positive root $\beta_j$ in $J$. Let $\alpha \in \Phi^+$, $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i$, and let $\mathrm{ht}(\alpha) = \sum_{i=1}^{\theta} n_i$, $g_\alpha = \prod g_i^{n_i}$, $\chi_\alpha = \prod \chi_i^{n_i}$. There is a unique connected component $J_\alpha$ of the Dynkin diagram of $\Phi$ for which $n_i \neq 0$ implies $i \in J_\alpha$. We write $J = J_\alpha$ when it is clear which $\alpha$ is intended. Choose scalars $(\mu_\alpha)_{\alpha\in\Phi^+}$, called *root vector parameters*, such that

$$\mu_\alpha = 0 \ \text{ if } \ g_\alpha^{N_\alpha} = 1 \ \text{ or } \ \chi_\alpha^{N_\alpha} \neq \varepsilon. \tag{5.0.12}$$

If $a = (a_1,\dots,a_p) \in \mathbb{N}^p - \{0\}$, define

$$\underline{a} := a_1\beta_1 + \cdots + a_p\beta_p.$$

In particular, letting $e_l := (\delta_{kl})_{1\le k\le p} \in \mathbb{N}^p - \{0\}$, we have $\underline{e_l} = \beta_l$.

The finite dimensional Hopf algebra $u(\mathcal{D},\lambda,\mu)$ is the quotient of $U(\mathcal{D},\lambda)$ by the ideal generated by all

$$x_\alpha^{N_\alpha} - u_\alpha(\mu) \qquad (\alpha \in \Phi^+) \tag{5.0.13}$$

where $u_\alpha(\mu) \in k\Gamma$ is defined inductively on $\Phi^+$ as follows [2, Defn. 2.14]. If $\alpha$ is a simple root, then $u_\alpha(\mu) := \mu_\alpha(1 - g_\alpha^{N_\alpha})$. If $\alpha$ is not simple, write $\alpha = \beta_l$ for some $l$, and then

$$u_\alpha(\mu) := \mu_\alpha(1 - g_\alpha^{N_\alpha}) + \sum_{\substack{b,c\in\mathbb{N}^p-\{0\}\\ \underline{b}+\underline{c}=\alpha}} t_{b,c}^{e_l}\mu_b u^c \tag{5.0.14}$$

where

(i) scalars $t^a_{b,c}$ are uniquely defined by

$$\begin{aligned}\Delta(x_{\beta_1}^{a_1N_1}\cdots x_{\beta_p}^{a_pN_p}) &= x_{\beta_1}^{a_1N_1}\cdots x_{\beta_p}^{a_pN_p}\otimes 1 + g_{\beta_1}^{a_1N_1}\cdots g_{\beta_p}^{a_pN_p}\otimes x_{\beta_1}^{a_1N_1}\cdots x_{\beta_p}^{a_pN_p}\\ &\quad + \sum_{\substack{b,c\in\mathbb{N}^p-\{0\}\\ \underline{b}+\underline{c}=\underline{a}}} t^a_{b,c}x_{\beta_1}^{b_1N_1}\cdots x_{\beta_p}^{b_pN_p}g_{\beta_1}^{c_1N_1}\cdots g_{\beta_p}^{c_pN_p}\otimes x_{\beta_1}^{c_1N_1}\cdots x_{\beta_p}^{c_pN_p} \quad \text{[2, Lemma 2.8];}\end{aligned}$$

(ii) scalars $\mu_a$ and elements $u^a \in k\Gamma$ are defined, via induction on $\mathrm{ht}(\underline{a})$, by the requirements that $\mu_{e_l} = \mu_{\beta_l}$ for $1 \le l \le p$, $\mu_a = 0$ if $g_{\beta_1}^{a_1N_1}\cdots g_{\beta_p}^{a_pN_p} = 1$,

and

$$u^a := \mu_a(1 - g_{\beta_1}^{a_1 N_1} \cdots g_{\beta_p}^{a_p N_p}) + \sum_{\substack{b,c \in \mathbb{N}^p - \{0\} \\ \underline{b} + \underline{c} = \underline{a}}} t_{b,c}^a \mu_b u^c,$$

where the remaining values of $\mu_a$ are determined by $u^a = u^r u^s$ where $a = (a_1, \ldots, a_l, 0, \ldots, 0)$, $a_l \geq 1$, $s = e_l$, and $a = r + s$ [2, Theorem 2.13].

Andruskiewitsch and Schneider give the elements $u_\alpha(\mu)$ explicitly in type $A_\bullet$ in [1, Theorem 6.1.8].

**Remark 5.0.15.** It follows from the induction [2, Theorem 2.13] that if $\alpha$ is a positive root of *smallest* height for which $\mu_\alpha \neq 0$, then $\mu_a = 0$ for all $a \in \mathbb{N}^p - \{0\}$ such that $\mathrm{ht}(\underline{a}) < \mathrm{ht}(\alpha)$.

The following theorem is [2, Classification Theorem 0.1].

**Theorem 5.0.16** (Andruskiewitsch-Schneider)**.** *Assume the field $k$ is algebraically closed and of characteristic 0. The Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ are finite dimensional and pointed. If $H$ is a finite dimensional pointed Hopf algebra having abelian group of grouplike elements with order not divisible by primes less than 11, then $H \simeq u(\mathcal{D}, \lambda, \mu)$ for some $\mathcal{D}$, $\lambda$, $\mu$.*

We will need a lemma about central grouplike elements and skew primitive elements.

**Lemma 5.0.17.** *Let $\alpha \in \Phi^+$ for which $\chi_\alpha^{N_\alpha} = \varepsilon$. Then*
(i) *$g_\alpha^{N_\alpha}$ is in the center of $U(\mathcal{D}, \lambda)$, and*
(ii) *there are no $(g_\alpha^{N_\alpha}, 1)$-skew primitives in $\bigoplus_{i \geq 1} u(\mathcal{D}, \lambda, \mu)_i$.*

*Proof.* (i) It suffices to prove that $g_\alpha^{N_\alpha}$ commutes with $x_j$ for each $j$. Note that $g_\alpha^{N_\alpha} x_j = \chi_j(g_\alpha^{N_\alpha}) x_j g_\alpha^{N_\alpha}$. Write $g_\alpha = \prod g_i^{n_i}$ where $\alpha = \sum_{i=1}^{\theta} n_i \alpha_i$. By the Cartan condition (5.0.5) and the hypothesis $\chi_\alpha^{N_\alpha} = \varepsilon$, we have

$$\chi_j(g_\alpha^{N_\alpha}) = \prod_{i=1}^{\theta} \chi_j(g_i^{n_i})^{N_\alpha} = \prod_{i=1}^{\theta} (\chi_i(g_i)^{a_{ij}} \chi_i^{-1}(g_j))^{n_i N_\alpha} = \chi_\alpha^{N_\alpha}(g_i^{a_{ij}} g_j^{-1}) = 1.$$

(ii) Each skew primitive in $u(\mathcal{D}, \lambda, \mu)$ is of degree at most 1, and the only $(g,1)$-skew primitives in degree 1, for any $g \in \Gamma$, are in the span of the $x_i$ [2, (5.5) and Cor. 5.2]. Now $g_i \neq g_\alpha^{N_\alpha}$ for each $i$, since the latter element is central by (i), while the former is not. $\square$

We remark that the special case $u(\mathcal{D}, 0, 0)$ is a graded bialgebra, the grading given by the coradical filtration. In this case, $u(\mathcal{D}, 0, 0) \simeq \mathcal{B}(V) \# k\Gamma$, the Radford biproduct (or bosonization) of the Nichols algebra $\mathcal{B}(V)$ of the Yetter-Drinfeld module $V$ over $k\Gamma$. For details, see [2, Cor. 5.2].

We wish to understand $u(\mathcal{D}, \lambda, \mu)$ as a graded bialgebra deformation of $u(\mathcal{D}, 0, 0)$. We now describe this graded case in more detail. Let $R = \mathcal{B}(V)$ be the subalgebra

of $u(\mathcal{D},0,0)$ generated by all $x_i$, and $\widetilde{R}$ the subalgebra of $U(\mathcal{D},0)$ generated by all $x_i$, so that $R \simeq \widetilde{R}/(x_\alpha^{N_\alpha} \mid \alpha \in \Phi^+)$. By [2, Thm. 2.6], $\widetilde{R}$ has PBW basis

$$x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} \qquad (a_1, \dots, a_p \geq 0), \tag{5.0.18}$$

and further,

$$[x_\alpha, x_\beta^{N_\beta}]_c = 0 \tag{5.0.19}$$

for all $\alpha, \beta \in \Phi^+$. Thus $R$ has PBW basis consisting of all elements in (5.0.18) for which $0 \leq a_i < N_i$. Choose the section of the quotient map $\pi : \widetilde{R} \to R$ for which the image of an element $r$ of $R$ is the unique element $\widetilde{r}$ that is a linear combination of the PBW basis elements of $\widetilde{R}$ with $a_i < N_i$ for all $i = 1, \dots, p$. This choice of section is used in Section 6.1 below.

## 6. Applications to some pointed Hopf algebras

We will apply the cohomological results of the first part of the paper to compute the degree 2 bialgebra cohomology of the Radford biproduct $R\#k\Gamma \simeq u(\mathcal{D},0,0)$ defined in Section 5. We then use the result to understand deformations.

6.1. **Hochschild cohomology of $u(\mathcal{D},0,0)$.** We first compute $\mathrm{H}_h^2(R,k)$ and then apply the isomorphism (2.4.1) to obtain $\mathrm{H}_h^2(B,k)$ where $B = R\#k\Gamma$. Hochschild one-cocycles on $R$ with coefficients in $k$ are simply derivations from $R$ to $k$, that is functions $f\colon R \to k$ such that $f(rs) = \varepsilon(r)f(s) + f(r)\varepsilon(s)$ for all $r, s \in R$. These may be identified with the linear functions from $R^+/(R^+)^2$ to $k$, where $R^+ = \ker\varepsilon$ is the augmentation ideal. A basis for the space of such functions is $\{f_i \mid 1 \leq i \leq \theta\}$, where for each $i$,

$$f_i(x_j) = \delta_{ij} \quad (1 \leq j \leq \theta).$$

All coboundaries in degree one are 0, and so $\{f_i \mid 1 \leq i \leq \theta\}$ may be identified with a basis of $\mathrm{H}_h^1(R,k)$. We obtain some elements of $\mathrm{H}_h^2(R,k)$ as cup products of pairs of the $f_i$: For $1 \leq i < j \leq \theta$, define linear maps on pairs of PBW basis elements (5.0.18), $\mathbf{x}^{\mathbf{a}} = x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}$ and $\mathbf{x}^{\mathbf{b}} = x_{\beta_1}^{b_1} \cdots x_{\beta_p}^{b_p}$:

$$f_{ji}(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}) = \begin{cases} 1, & \text{if } \mathbf{x}^{\mathbf{a}} = x_j \text{ and } \mathbf{x}^{\mathbf{b}} = x_i \\ 0, & \text{otherwise.} \end{cases} \tag{6.1.1}$$

Then $f_{ji} = f_j \smile f_i$.

Other Hochschild two-cocycles of $R$, with coefficients in $k$, are indexed by the positive roots $\Phi^+$ and defined as follows: Recall from the end of Section 5 that $\widetilde{R}$ is an algebra for which $R \simeq \widetilde{R}/(x_\alpha^{N_\alpha} \mid \alpha \in \Phi^+)$. Let $\widetilde{R}^+$ be the augmentation ideal of $\widetilde{R}$. For each $\alpha \in \Phi^+$, define $\widetilde{f}_\alpha : \widetilde{R}^+ \otimes \widetilde{R}^+ \to k$ by

$$\widetilde{f}_\alpha(r,s) = \gamma_{(0,\dots,0,N_\alpha,0,\dots,0)}$$

where $N_\alpha$ is in the $i$th position if $\alpha = \beta_i$, and $rs = \sum_{\mathbf{a}} \gamma_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ in $\widetilde{R}$. By its definition, $\widetilde{f}_\alpha$ is associative on $\widetilde{R}^+$, so it may be extended (trivially) to a normalized Hochschild two-cocycle on $\widetilde{R}$. In fact $\widetilde{f}_\alpha$ is a coboundary on $\widetilde{R}$: $\widetilde{f}_\alpha = \partial h_\alpha$ where $h_\alpha(r)$ is the coefficient of $x_\alpha^{N_\alpha}$ in $r \in \widetilde{R}$ written as a linear combination of PBW basis elements. We next show that $\widetilde{f}_\alpha$ factors through the quotient map $\pi : \widetilde{R} \to R$ to give a Hochschild two-cocycle $f_\alpha$ on $R$, and that $f_\alpha$ is *not* a coboundary on $R$. We must show that $\widetilde{f}_\alpha(r, s) = 0$ whenever either $r$ or $s$ is in the kernel of the quotient map $\pi : \widetilde{R}^+ \to R^+$. It suffices to prove this for PBW basis elements. Suppose $\mathbf{x}^{\mathbf{a}} \in \ker \pi$. That is, $a_j \geq N_j$ for some $j$. Write $\mathbf{x}^{\mathbf{a}} = \kappa x_{\beta_j}^{N_j} \mathbf{x}^{\mathbf{b}}$ where $\kappa$ is a nonzero scalar and $\mathbf{b}$ may be 0; note this is possible by the relation (5.0.19). Then $\widetilde{f}_\alpha(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{c}}) = \kappa \widetilde{f}_\alpha(x_{\beta_j}^{N_j} \mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}})$, and this is the coefficient of $x_\alpha^{N_\alpha}$ in the product $\kappa x_{\beta_j}^{N_j} \mathbf{x}^{\mathbf{b}} \mathbf{x}^{\mathbf{c}}$. However, the coefficient of $x_\alpha^{N_\alpha}$ is 0: If $\alpha = \beta_i$ and $j = i$, then since $\mathbf{x}^{\mathbf{c}} \in \widetilde{R}^+$, this product cannot have a nonzero coefficient for $x_\alpha^{N_\alpha}$. If $j \neq i$, the same is true since $x_{\beta_j}^{N_j}$ is a factor of $\mathbf{x}^{\mathbf{a}} \mathbf{x}^{\mathbf{c}}$. A similar argument applies to $\widetilde{f}_\alpha(\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{c}})$ if $\mathbf{x}^{\mathbf{c}} \in \ker \pi$.

Thus $\widetilde{f}_\alpha$ factors through $\pi \colon \widetilde{R} \to R$, and we may define $f_\alpha \colon R^+ \otimes R^+ \to k$ by

$$f_\alpha(r, s) = \widetilde{f}_\alpha(\widetilde{r}, \widetilde{s}), \tag{6.1.2}$$

where $\widetilde{r}, \widetilde{s}$ are defined via the section of $\pi$ chosen at the end of Section 5. We must verify that $f_\alpha$ is associative on $R^+$. Let $r, s, u \in R^+$. Since $\pi$ is an algebra homomorphism, we have $\widetilde{r} \cdot \widetilde{s} = \widetilde{rs} + y$ and $\widetilde{s} \cdot \widetilde{u} = \widetilde{su} + z$ for some elements $y, z \in \ker \pi$. Since $\ker \pi \otimes \widetilde{R} + \widetilde{R} \otimes \ker \pi \subset \ker \widetilde{f}_\alpha$, we have

$$\begin{aligned} f_\alpha(rs, u) = \widetilde{f}_\alpha(\widetilde{rs}, \widetilde{u}) &= \widetilde{f}_\alpha(\widetilde{r} \cdot \widetilde{s} - y, \widetilde{u}) \\ &= \widetilde{f}_\alpha(\widetilde{r} \cdot \widetilde{s}, \widetilde{u}) \\ &= \widetilde{f}_\alpha(\widetilde{r}, \widetilde{s} \cdot \widetilde{u}) = \widetilde{f}_\alpha(\widetilde{r}, \widetilde{su}) = f_\alpha(r, su). \end{aligned}$$

As we will see, we only need the functions $f_{ji}$ when $i \not\sim j$, that is $i$ and $j$ are in different connected components of the Dynkin diagram of $\Phi$. Together with the $f_\alpha$, $\alpha \in \Phi^+$, these represent a linearly independent subset of $\mathrm{H}^2_h(R, k)$:

**Theorem 6.1.3.** *The set* $\{f_\alpha \mid \alpha \in \Phi^+\} \cup \{f_{ji} \mid 1 \leq i < j \leq \theta,\ i \not\sim j\}$ *represents a linearly independent subset of* $\mathrm{H}^2_h(R, k)$.

*Proof.* Let

$$f = \sum_{\alpha \in \Phi^+} c_\alpha f_\alpha + \sum_{\substack{1 \leq i < j \leq \theta \\ i \not\sim j}} c_{ji} f_{ji}$$

for scalars $c_\alpha, c_{ji}$. Assume $f = \partial h$ for some $h : R \to k$. Then for each $\alpha \in \Phi^+$,

$$c_\alpha = f(x_\alpha, x_\alpha^{N_\alpha - 1}) = \partial h(x_\alpha, x_\alpha^{N_\alpha - 1}) = -h(x_\alpha^{N_\alpha}) = 0$$

as $x_\alpha, x_\alpha^{N_\alpha-1} \in R^+$ and $x_\alpha^{N_\alpha} = 0$ in $R$. For each pair $i, j$ $(1 \le i < j \le \theta,\ i \not\sim j)$, $x_j x_i = \chi_i(g_j) x_i x_j$ since $i \not\sim j$, and so

$$\begin{aligned} c_{ji} \ = \ f(x_j, x_i) \ &= \ \partial h(x_j, x_i) \\ &= \ -h(x_j x_i) \\ &= \ -h(\chi_i(g_j) x_i x_j) \ = \ \chi_i(g_j) f(x_i, x_j) \ = \ 0. \end{aligned}$$

□

Due to the isomorphism (2.4.1), we are primarily interested in $\Gamma$-invariant Hochschild two-cocycles from $R \otimes R$ to $k$. The action of $\Gamma$ on the functions $f_\alpha, f_{ji}$ is diagonal, and so we determine those $f_\alpha, f_{ji}$ that are themselves $\Gamma$-invariant.

**Theorem 6.1.4.** *If* $|\Gamma|$ *is not divisible by primes less than* 11*, then*

$$\{f_\alpha, f_{ji}\}^\Gamma := \{f_\alpha \mid \alpha \in \Phi^+,\ \chi_\alpha^{N_\alpha} = \varepsilon\} \cup \{f_{ji} \mid 1 \le i < j \le \theta,\ i \not\sim j,\ \chi_i\chi_j = \varepsilon\}$$

*is a basis of* $\mathrm{H}^2_h(R\#k\Gamma, k)$. *In particular, if* $l \le -3$*, then* $\{f_\alpha | \chi_\alpha^{N_\alpha} = \varepsilon,\ N_\alpha \operatorname{ht}(\alpha) = -l\}$ *is a basis for* $\mathrm{H}^2_h(R\#k\Gamma, k)_l$.

**Remark 6.1.5.** The condition $i \not\sim j$ is implied by the condition $\chi_i\chi_j = \varepsilon$, and so is redundant: A proof of this fact consists of a case-by-case analysis using the Cartan condition (5.0.5) for the pairs $i, j$ and $j, i$. We put the condition $i \not\sim j$ in the statement of the theorem for clarity.

*Proof.* The action of $\Gamma$ on $\mathrm{H}^2_h(R, k)$ comes from the dual action of $\Gamma$ on $R \otimes R$, that is, $(g \cdot f)(r, s) = f(g^{-1} \cdot r, g^{-1} \cdot s)$. Therefore $g \cdot f_\alpha = \chi_\alpha^{-N_\alpha}(g) f_\alpha$ and $g \cdot f_{ji} = \chi_i^{-1}(g)\chi_j^{-1}(g) f_{ji}$. Thus the subset of those functions from Theorem 6.1.3 that are $\Gamma$-invariant consists of the $f_\alpha$ for which $\chi_\alpha^{N_\alpha} = \varepsilon$ and the $f_{ji}$ for which $\chi_i\chi_j = \varepsilon$. By Remark 6.1.5, this proves that the given set represents a linearly independent subset of $\mathrm{H}^2_h(R\#k\Gamma, k)$.

The fact that the given set spans $\mathrm{H}^2_h(R\#k\Gamma, k)$ is a consequence of [9, Lemma 5.4]: That lemma states in our case that $\mathrm{H}^2(R, k)$ (equivalently, $\mathrm{H}^2_h(R, k)$) has basis in one-to-one correspondence with that of $I/(T^+(V) \cdot I + I \cdot T^+(V))$, where $R = T(V)/I$ ($I$ is the ideal of relations). This follows by looking at the minimal resolution of $k$. As a consequence, $\mathrm{H}^2_h(R\#k\Gamma, k)$, which is isomorphic to $\mathrm{H}^2_h(R, k)^\Gamma$, has basis in one-to-one correspondence with that of the $\Gamma$-invariant subspace of $I/(T^+(V) \cdot I + I \cdot T^+(V))$. The $\Gamma$-invariant root vector relations and linking relations give rise to the elements $f_\alpha$ and $f_{ji}$ in the statement of the theorem. It remains to show that the Serre relations do not give rise to $\Gamma$-invariant elements in cohomology. If the Serre relation (5.0.9) did give rise to a $\Gamma$-invariant element in cohomology, then $\chi_i^{1-a_{ij}}\chi_j = \varepsilon$ by considering the $\Gamma$-action. This is not possible: The Cartan condition for the pairs $i, j$ and $j, i$, together with this equation, implies $\chi_i(g_i)^{a_{ij}+a_{ji}-a_{ij}a_{ji}} = 1$. A case-by-case analysis shows that this implies $\chi_i(g_i)$ has order 3, 5, or 7, contradicting our assumption that the order of $\Gamma$ is not divisible by primes less than 11.

The last statement of the theorem is now immediate from the observation that $f_\alpha$ is homogeneous of degree $-N_\alpha \operatorname{ht}(\alpha)$, and $f_{ji}$ is homogeneous of degree $-2$. $\square$

**Remark 6.1.6.** Masuoka independently obtained a proof that the given set spans $\mathrm{H}^2_h(R\#k\Gamma, k)$, using completely different methods and results from his preprint [16].

Note that the conditions in the theorem on the $\chi_\alpha$ and $\chi_i$ are "half" of the conditions (5.0.6) and (5.0.12) under which nontrivial linking or root vector relations may occur. The other half of those conditions, involving elements of $\Gamma$, will appear after we apply the formula (3.4.1) to obtain corresponding *bialgebra* two-cocycles. (The bialgebra two-cocycle will be 0 when the condition on the appropriate group element is not met.)

6.2. **Bialgebra two-cocycles.** Let $B = R\#k\Gamma$ as before. We wish to apply the connecting homomorphism in the long exact sequence (3.3.1) to elements of $\mathrm{H}^2_h(B, k)$ from Theorem 6.1.4, in order to obtain bialgebra two-cocycles. First we prove that the connecting homomorphism is surjective.

**Theorem 6.2.1.** *Assume the order of $\Gamma$ is not divisible by 2 or 3. If $\mathrm{H}^2_h(B, k) = \operatorname{Span}\{f_{ji}, f_\alpha\}^\Gamma$ and $l < 0$, then the connecting homomorphism $\delta\colon \mathrm{H}^2_h(B, k)_l \to \widehat{\mathrm{H}}^2_b(B)_l$ is surjective.*

*Proof.* Let $M$ be as in (4.1.1). Note that the fact that $\mathrm{H}^2_h(B, k) = \operatorname{Span}\{f_{ji}, f_\alpha\}^\Gamma$ translates into the fact that $\{x_j \otimes x_i - \chi_j(g_i)x_i \otimes x_j | \chi_i\chi_j\} \bigcup \{x_\alpha \otimes x_\alpha^{N_\alpha - 1} | \chi_\alpha^{N_\alpha} = \varepsilon\}$ forms a basis for $M^\Gamma$. Hence, if $M_{h,\varepsilon,r} \neq 0$, then either $h = g_i g_j$ with $\chi_i\chi_j = \varepsilon$, or $h = g_\alpha^{N_\alpha}$ with $\chi_\alpha^{N_\alpha} = \varepsilon$. If $h = g_\alpha^{N_\alpha}$, then there are no $(1, h)$-primitives in of positive degree in $B$ by Lemma 5.0.17(ii). We now show that there are no $(1, h)$-primitives of positive degree in $B$ (equivalently $R_1$) whenever $h = g_i g_j$. Suppose otherwise, i.e., for some $k$ we have $g_k = g_i g_j$. Since $i \not\sim j$, we also have that either $i \not\sim k$, or $j \not\sim k$. Without loss of generality suppose the latter. Then $\chi_k(g_k) = \chi_k(g_i g_j)\chi_i(g_k)\chi_j(g_k) = (\chi_k(g_i)\chi_j(g_k))(\chi_k(g_j)\chi_j(g_k)) = \chi_k(g_k)^{a_{ki}}$ and hence $\chi_k(g_k)^{1-a_{ki}} = 1$. This is impossible, since $1 - a_{ki} \in \{1, 2, 3, 4\}$ and $|\Gamma|$ is not divisible by 2 or 3.

Hence the conditions of Theorem 4.2.3 are satisfied. $\square$

**Remark 6.2.2.** Assume that the order of $\Gamma$ is not divisible by 2 or 3. Additionally assume that the order of $\Gamma$ is not divisible by 5 whenever the Dynkin diagram associated to $\mathcal{D}$ contains a copy of $B_n$ with $n \geq 3$. A similar case by case analysis to the one in the proof of Theorem 6.1.4 shows that there are no $(1, h)$-primitives whenever $h = g_i^{1-a_{ij}} g_j$ with $\chi_i^{1-a_{ij}}\chi_j = \varepsilon$. This observation can then be used to show that the connecting homomorphism $\delta\colon \mathrm{H}^2_h(B, k)_l \to \widehat{\mathrm{H}}^2_b(B)_l$, where $B = u(\mathcal{D}, 0, 0)$ and $l < 0$ is surjective.

Now let $f\colon B\otimes B \to k$ be a Hochschild two-cocycle. The formula (3.4.1) applied to $f$ yields

$$F(a,b) = f(a_1,b_1)a_2b_2 - f(a_2,b_2)a_1b_1, \tag{6.2.3}$$

a bialgebra two-cocycle representing an element in $\widehat{\mathrm{H}}^2_b(B,B)$. We apply this formula to $f = f_{ji}, f_\alpha$, defined in (6.1.1) and (6.1.2), to obtain explicit bialgebra two-cocycles $F = F_{ji}, F_\alpha$. For our purposes, it will suffice to compute the value of each $F_{ji}, F_\alpha$ on a single well-chosen pair of elements in $R$. In order to compute them on arbitrary pairs of elements of $B$, one must use (6.2.3) and Lemma 2.4.2.

**Lemma 6.2.4.** *If $f\colon B\otimes B \to k$ is a homogeneous $\varepsilon$-cocycle of degree $l < 0$, $\delta f = (F,0)$, and $x \in R_i$, $y \in R_j$, with $i + j = -l$, are PBW-basis elements in components $g_x$, $g_y$, then*

$$F(x,y) = f(x,y)(1 - g_xg_y).$$

*In particular*

$$F_{ji}(x_j,x_i) = 1 - g_jg_i, \tag{6.2.5}$$

*and*

$$F_\alpha(x_\alpha, x_\alpha^{N_\alpha-1}) = 1 - g_\alpha^{N_\alpha}. \tag{6.2.6}$$

*Proof.* The hypotheses imply that $\Delta x = x\otimes 1 + g_x\otimes x + u$ and $\Delta y = y\otimes 1 + g_y\otimes y + v$, where $u = \sum_r u'_r\otimes u''_r \in \bigoplus_{p=1}^{i-1} B_p\otimes B_{i-p}$ and $v = \sum_s v'_s\otimes v''_s \in \bigoplus_{q=1}^{j-1} B_q\otimes B_{j-q}$. Then $f(u'_r,v'_s) = 0 = f(u''_r,v''_s)$ due to degree considerations and hence $F(x,y) = f(x,y)1 - f(x,y)g_xg_y$. $\square$

Note that $F_{ji}(x_j,x_i) = 0$ exactly when $g_ig_j = 1$, and $F_\alpha(x_\alpha, x_\alpha^{N_\alpha-1}) = 0$ exactly when $g_\alpha^{N_\alpha} = 1$. More generally, if $x, y$ are PBW-basis elements of joint degree 2 (resp. $\mathrm{ht}(\alpha)N_\alpha$), then $F_{ji}(x,y) \in k(1 - g_jg_i)$ (resp. $F_\alpha(x,y) \in k(1 - g_\alpha^{N_\alpha})$).

Combined with the conditions on $\chi_\alpha$ and $\chi_i$ in Theorem 6.1.4, we have recovered precisely the conditions in (5.0.6) and (5.0.12) under which there exist nontrivial linking and root vector relations. In Theorem 6.3.1 below, we make the connection between these bialgebra two-cocycles and the pointed Hopf algebras $u(\mathcal{D},\lambda,\mu)$.

Our calculations above, combined with Theorems 6.1.4 and 6.2.1 now allow us to determine completely $\widehat{\mathrm{H}}^2_b(B)_l$, $l < 0$, for coradically graded Hopf algebras $B = R\#\Gamma$ in the Andruskiewitsch-Schneider program. We have the following theorem.

**Theorem 6.2.7.** *Let $B = u(\mathcal{D},0,0)$ and assume that $|\Gamma|$ is not divisible by 2 or 3. If $\mathrm{H}^2_h(B,k) = \mathrm{Span}\{f_{ji}, f_\alpha\}^\Gamma$ (see for example Theorem 6.1.4), then the set*

$$\big\{(F_\alpha,0),(F_{j,i},0)\,|\,\alpha\in\Phi^+, 1\le i<j\le\theta, g_\alpha^{N_\alpha}\ne 1, \chi_\alpha^{N_\alpha}=\varepsilon, i\not\sim j, g_ig_j\ne 1, \chi_i\chi_j=\varepsilon\big\}$$

*is a basis for $\bigoplus_{l<0}\widehat{\mathrm{H}}^2_b(B)_l$.*

*Proof.* $F_\alpha$ is a homogeneous cocycle of degree $l := -\operatorname{ht}(\alpha)N_\alpha$. If $g_\alpha^{N_\alpha} = 1$, then $(F_\alpha)_{-l} = 0$ and hence by Lemma 4.2.2 (ii) $(F_\alpha, 0)$ is a coboundary. We can show that if $g_i g_j = 1$, then $(F_{i,j}, 0)$ is a coboundary in a similar fashion. Hence by Theorems 6.1.4 and 6.2.1 and our above calculations, the given set spans $\bigoplus_{l<0} \widehat{\mathrm{H}}_b^2(B)_l$.

Note that it is sufficient to show that for every $l < 0$, the Hochschild cohomology classes of cocycles $F_\alpha, F_{i,j}$ of degree $l$ are linearly independent. This is achieved as follows.

If $l < -2$, then $\left(\sum_{\operatorname{ht}(\alpha)N_\alpha=-l} \lambda_\alpha F_\alpha\right)(x_\alpha, x_\alpha^{N_\alpha-1}) = \lambda_\alpha(1 - g_\alpha^{N_\alpha})$, but for $s\colon B \to B$ homogeneous of degree $l$, we have $\partial^h(s)(x_\alpha, x_\alpha^{N_\alpha-1}) = x_\alpha s(x_\alpha^{N_\alpha-1}) - s(x_\alpha^{N_\alpha}) + s(x_\alpha)x_\alpha^{N_\alpha-1} = 0$.

If $l = -2$, then $\left(\sum_{i,j} \lambda_{ji} F_{ji}\right)(x_j \otimes x_i - q_{i,j} x_i \otimes x_j) = \lambda_{i,j}(1 - g_i g_j)$ and if $s\colon B \to B$ is homogeneous of degree $-2$, then $\partial^h(s)(x_j \otimes x_i - \chi_j(g_i) x_i \otimes x_j) = 0$. $\square$

**Remark 6.2.8.** For positive $l$ one can use the homomorphism $\delta\colon H_c^2(k, B)_l \to \widehat{\mathrm{H}}_b^2(B)_l$ to obtain a similar description for $\bigoplus_{l>0} \widehat{\mathrm{H}}_b^2(B)_l$. However, the positive part of the truncated bialgebra cohomology is not relevant in the context of graded bialgebra deformations.

6.3. **Graded bialgebra deformations.** Now let $B = u(\mathcal{D}, \lambda, \mu)$, defined in Section 5. Also assume that the order of $\Gamma$ is not divisible by primes $< 11$. These Hopf algebras are in general filtered by the coradical filtration, with $\deg(x_i) = 1$ ($i = 1, \ldots, \theta$) and $\deg(g) = 0$ ($g \in \Gamma$). The filtration allows us to define related Hopf algebras over $k[t]$, where $t$ is an indeterminate, as in [6]: By [2, Theorem 3.3(1)], $B$ has PBW basis $\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} g \mid 1 \le a_i < N_i,\ g \in \Gamma\}$. Express each element of $B$ uniquely as a linear combination of these basis elements. Then there exist unique maps $m_s : B \otimes B \to B$, homogeneous of degree $-s$, such that

$$m(a \otimes b) = \sum_{s \ge 0} m_s(a \otimes b)$$

for all $a, b \in H$. Now define a new multiplication $m_t : B \otimes B \to B[t]$ by

$$m_t(a \otimes b) = \sum_{s \ge 0} m_s(a \otimes b) t^s,$$

and extend $k[t]$-linearly to $B[t] \otimes_{k[t]} B[t]$. In particular the analogs of the linking and root vector relations (5.0.10) and (5.0.13) for $B[t]$ will now involve powers of $t$. When we write $B[t]$, we will always mean the vector space $B[t]$ with multiplication $m_t$ and the usual (graded) comultiplication. In this way $B[t]$ is a graded deformation of $\operatorname{Gr} B = u(\mathcal{D}, 0, 0)$.

If $s > 0$, then define $\lambda^{(s)}$ and $\mu^{(s)}$ by

$$\lambda^{(s)} = \begin{cases} \lambda; & \text{if } s = 2 \\ 0; & \text{otherwise} \end{cases}, \quad \mu_\alpha^{(s)} = \begin{cases} \mu_\alpha; & \text{if } s = \operatorname{ht}(\alpha)N_\alpha \\ 0; & \text{otherwise} \end{cases}.$$

Note that $u(\mathcal{D},\mu,\lambda)[t]/(t^{s+1}) \simeq u(\mathcal{D},\mu^{(s)},\lambda^{(s)})[t]/(t^{s+1})$.

Let $r = r(\lambda,\mu)$ be the smallest positive integer, if it exists, such that $m_r \neq 0$. (If it does not exist, set $r = r(\lambda,\mu) = 0$.) Since $B[t]$ is a bialgebra, $(m_r, 0)$ is necessarily a bialgebra two-cocycle, of degree $-r$, and $B[t]/(t^{r+1})$ is an $r$-deformation (see Section 2.2). Note that if $\lambda \neq 0$, then $r = 2$, due to the linking relations (5.0.10). Recall the definitions (5.0.7) of $\lambda_{ji}$ $(i < j)$ and of $F_{ji}, F_\alpha$ via (6.2.3) above. The following is a nice consequence of Theorem 6.2.7.

**Theorem 6.3.1.**

(i) *Let $B[t] = u(\mathcal{D},\lambda,\mu)[t]$ and $B'[t] = u(\mathcal{D},\lambda',\mu')[t]$ and let $s$ be a nonnegative integer such that $\lambda^{(s)} = \lambda'^{(s)}$ and $\mu^{(s)} = \mu'^{(s)}$. Then $B[t]/(t^{s+2})$ and $B'[t]/(t^{s+2})$ are $(s+1)$-deformations of $u(\mathcal{D},0,0)$ extending the same $s$-deformation $B[t]/(t^{s+1}) = B'[t]/(t^{s+1})$. If $m_t = m + tm_1 + \ldots t^s m_s + t^{s+1}m_{s+1} + \ldots$ then $m'_t = m + tm_1 + \ldots\ldots t^s m_s + t^{s+1}m'_{s+1} + \ldots$ and $(m_{s+1} - m'_{s+1}, 0)$ is a bialgebra cocycle of degree $-(s+1)$ cohomologous to $(F,0)$, where*

$$F = \sum_{1\le i<j\le\theta} \delta_{s+1,2}(\lambda_{ji} - \lambda'_{ji})F_{ji} + \sum_{\alpha\in\Phi^+} \delta_{s+1,\mathrm{ht}(\alpha)N_\alpha}(\mu_\alpha - \mu'_\alpha)F_\alpha.$$

(ii) *In particular, the Hopf algebra $u(\mathcal{D},\lambda,\mu)[t]$ is a graded deformation of $u(\mathcal{D},0,0) = R\#k\Gamma$, over $k[t]$, with infinitesimal deformation $(m_r,0)$ cohomologous to $(F,0)$, where*

$$F = \sum_{1\le i<j\le\theta} \delta_{r,2}\lambda_{ji}F_{ji} + \sum_{\alpha\in\Phi^+} \delta_{r,\mathrm{ht}(\alpha)N_\alpha}\mu_\alpha F_\alpha.$$

*Proof.* In view of Theorem 6.2.7 and Remark 2.2.1 it is clear that $(m_{s+1} - m'_{s+1}, 0)$ is a bialgebra cocycle cohomologous to $(F,0)$, where $F = \sum a_{ji}f_{ji} + \sum b_\alpha f_\alpha$ for some scalars $a_{ji}, b_\alpha$. Evaluating $m_{s+1}$ and $m'_{s+1}$ at $x_i\otimes x_j$ (if $s=1$) and $x_\alpha \otimes x_\alpha^{N_\alpha-1}$ (if $s = \mathrm{ht}(\alpha)N_\alpha - 1$) then identifies these coefficients. $\square$

We give one more class of examples to which our cohomological techniques apply, the rank one Hopf algebras of Krop and Radford [14]. Assume $k$ has characteristic 0. Let $\theta = 1$ and $(a_{11}) = (2)$. Let $\Gamma$ be a finite group (not necessarily abelian), $a = g_1$ a central element of $\Gamma$, and $\chi \in \widehat{\Gamma}$. Let $N$ be the order of $\chi(a)$. Let $x = x_1$, and $R = k[x]/(x^N)$, on which $\Gamma$ acts via $\chi$, that is $g\cdot x = \chi(g)x$ for all $g \in \Gamma$. Let $B = R\#k\Gamma$, a generalized Taft algebra with $\Delta(x) = x\otimes 1 + a\otimes x$. Similar to the functions $f_\alpha$ in Section 4.1, there is a Hochschild two-cocycle $f : R\otimes R \to k$ defined by

$$f(x^i,x^j) = \begin{cases} 1 & \text{if } i+j = N \\ 0 & \text{otherwise.}\end{cases}$$

This cocycle is $\Gamma$-invariant precisely when $\chi^N = \varepsilon$. In this case, let $\mu \in k$. There is a bialgebra deformation of $B$ in which the relation $x^N = 0$ is deformed to

$x^N = \mu(1-a^N)$; this is the Hopf algebra $H_{\mathcal{D}}$ of Krop and Radford [14]. In case $\Gamma$ is abelian, this example is included in the Andruskiewitsch-Schneider classification.

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